Chapter 7: Linear Dependence & Independence

In the last chapter, we saw that for a finite number of vectors like $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$, we can define their span to be the set of "all linear combinations" of them as:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m \mid a_1, a_2, \dots, a_m \in \mathbb{R}\}$$

Therefore, in spaces like $\mathbb{R}^2$ and $\mathbb{R}^3$ (vector geometry):

- $\operatorname{Span}\{\vec{0}\} = \{\vec{0}\}$
- If $\vec{v} \neq \vec{0}$, then $\operatorname{Span}\{\vec{v}\}$ is the line through origin with direction $\vec{v}$.
- If $\vec{u}, \vec{v} \neq \vec{0}$ and $\vec{u}$ & $\vec{v}$ are not parallel, then $\operatorname{span}\{\vec{u}, \vec{v}\}$ is a plane.

* Difficulties with span:

(I) Non-uniqueness of spanning sets:

Examples: $-$ $\operatorname{Span}\{(1,2)\} = \operatorname{Span}\{(2,4)\}$: parallel vectors and so span the same line.

$-$ $W = \operatorname{Span}\{(1,0,1), (0,1,0)\} = \operatorname{Span}\{(1,1,1), (1,-1,1)\} = U$

$(1,1,1) = (1,0,1) + (0,1,0)$, $(1,-1,1) = (1,0,1) - (0,1,0)$

$\Rightarrow U \subseteq W$ $\left((1,1,1) \text{ and } (1,-1,1) \in \operatorname{Span}\{(1,0,1), (0,1,0)\} \right)$

$(1,0,1) = \frac{1}{2}(1,1,1) + \frac{1}{2}(1,-1,1)$, $(0,1,0) = \frac{1}{2}(1,1,1) - \frac{1}{2}(1,-1,1)$

$\Rightarrow W \subseteq U$ $\left((1,0,1) \text{ and } (0,1,0) \in \operatorname{Span}\{(1,1,1), (1,-1,1)\} \right)$

So, $U = W$.

Note: Every vector space (except the zero vector space $\{\vec{0}\}$) has finitely many spanning sets and we cannot distinguish two subspaces by just looking at their spanning sets.

(II) More vectors in the spanning set does not imply more vectors in their span:

Example:

$$\text{span}\{(1,2)\} = \text{span}\{(0,0),(1,2)\} = \text{span}\{(1,2),(2,4),(3,6)\}$$

$$\text{span}\{(0,0),(1,2)\} = \{a(0,0)+b(1,2) \mid a,b \in \mathbb{R}\}$$
$$= \{b(1,2) \mid b \in \mathbb{R}\} = \text{span}\{(1,2)\}$$

$$\text{span}\{(1,2),(2,4),(3,6)\} = \{a(1,2)+b(2,4)+c(3,6) \mid a,b,c \in \mathbb{R}\}$$
$$= \{a(1,2)+2b(1,2)+3c(1,2) \mid a,b,c \in \mathbb{R}\}$$
$$= \{\underbrace{(a+2b+3c)}(1,2) \mid a,b,c \in \mathbb{R}\}$$
$$= \{d(1,2) \mid d \in \mathbb{R}\} = \text{span}\{(1,2)\}$$

Geometrically, the problem is that the vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ are either collinear (lying on one line), or coplanar (lying on one plane).

Question: what are the algebraic versions of collinear and coplanar vectors?

* Two vectors $\vec{u}$ and $\vec{v}$ are "collinear" if there exist scalars $a, b \in \mathbb{R}$ (not both zero), such that

$$\underline{a\vec{u} + b\vec{v} = \vec{0}}$$

example: $(1,2,1)$ and $(2,4,2)$ are collinear, since

$$2(1,2,1) + (-1)(2,4,2) = (0,0,0)$$

* Three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ are coplanar (lying on one plane) if there exist scalars $a, b, c \in \mathbb{R}$, (not all zero) such that

$$\underline{a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}}$$

example: $(1,0,1)$, $(0,1,0)$, $(1,1,1)$ are coplanar, since

$$(1,0,1) + (0,1,0) - (1,1,1) = (0,0,0)$$

* <u>Linear Dependence</u> (algebraic generalization of collinear & coplanar).

<u>Definition</u>: Let V be a vector space and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in V$. Then the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is "linearly dependent" (or we say that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are "linearly dependent") if and only if there are scalars $a_1, \dots, a_m \in \mathbb{R}$ <u>and not all zero</u>, such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m = \vec{0}$$

(It basically says that there exists a non-trivial solution to the dependence equation above)

So far:

- Two vectors $\vec{u}$ and $\vec{v}$ are not collinear (not parallel) if and only if

$$a\vec{u} + b\vec{v} = \vec{0} \Longleftrightarrow a = b = 0 \qquad a=b=0$$

(The only solution to the equation $a\vec{u}+b\vec{v}=\vec{0}$ is the trivial solution)

- Three vectors $\vec{u}, \vec{v}, \vec{w}$ are not coplanar if and only if

$$a\vec{u} + b\vec{v} + c\vec{w} \Longleftrightarrow a = b = c = 0$$

(The only solution to the equation $a\vec{u}+b\vec{v}+c\vec{w}=\vec{0}$ is the trivial solution $a=b=c=0$).

Definition: Let V be a vector space and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in V$. Then the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is "linearly independent" (or $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are in fact linearly independent) if and only if the "only solution" to the linear equation

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m = \vec{0}$$

is the trivial solution $a_1 = 0, \dots, a_m = 0$.

* If some vectors are not linear dependent, then they are linearly independent (and vice versa).

Examples:

(1) $\{(1,0), (0,1)\}$ are LI (linearly independent):

$$a(1,0) + b(0,1) = (0,0), \text{ then } (a,b) = (0,0)$$

$$\rightarrow \boxed{a=0 \text{ and } b=0} \checkmark$$

(2) $\{(1,1),(1,-1)\}$ are LI:

$$a(1,1)+b(1,-1)=(0,0) \Rightarrow (a+b,a-b)=(0,0)$$

$$\Rightarrow a=-b \text{ and } a=b \Rightarrow \boxed{a=b=0} \checkmark$$

(3) $\{(1,0),(0,1),(1,1)\}$ are LD:

$$a(1,0)+b(0,1)+c(1,1)=(0,0) \Rightarrow \text{If } a=b=1 \text{ and } c=-1$$

$$\Rightarrow 1(1,0)+1(0,1)-1(1,1)=(0,0)$$

we found a non-trivial solution : $a=b=1$, $c=-1 \Rightarrow$ LD.

(4) $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}$ are LD:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\0&1\end{pmatrix}-\begin{pmatrix}1&0\\0&-1\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

(5) $\{1, x, x^2\}$ are LI in $\mathbb{P}_2$ (Space of degree 2 polynomial functions):

$$a(1)+b(x)+c(x^2)=\vec{0}=0+0x+0x^2$$

$$\Rightarrow a=b=c=0.$$

(6) $\{4+4x+x^2, 1+x, x^2\}$ are LD in $\mathbb{P}_2$:

$$1(4+4x+x^2)+(-4)(1+x)+(-1)x^2=0 \quad \text{(non-trivial solution)}$$

(7) $\{1, \sin x, \cos x\}$ are LI in $F(\mathbb{R})$:

$$a(1)+b\sin x+c\cos x=0 \Rightarrow a=b=c=0$$

since,
$$\left.\begin{aligned} x=0 &\Rightarrow a+c=0 \\ x=\pi/2 &\Rightarrow a+b=0 \\ x=\pi &\Rightarrow a-c=0 \end{aligned}\right\} : (a=b=c=0) \checkmark$$

Some facts about LI & LD:

(1) If $\vec{v} \in V$, then $\{\vec{v}\}$ is LI if and only if $\vec{v} \neq \vec{0}$.

proof: if $\vec{v} \neq \vec{0}$, then $k\vec{v} = \vec{0}$ only has trivial solution $k=0$ (LI); if $\vec{v} = \vec{0}$, then $k\vec{v} = \vec{0}$ for non-trivial solutions (LD).

(2) If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ are LD, then any set containing $\vec{v}_1, \ldots, \vec{v}_m$ is also LD.

proof: $\vec{v}_1, \ldots, \vec{v}_m$ are LD, then $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$ has a non-trivial dependence solution, with not all a_is to be zero.

Then, for a larger set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, \vec{u}_1, \ldots, \vec{u}_k\}$, we have

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m + 0\vec{u}_1 + \cdots + 0\vec{u}_k = \vec{0},$$

where not all the coefficients are zeros. Hence, LD

(3) If $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is LI, then any subset is also LI.

(4) $\{\vec{0}\}$ is LD.

(5) Any set containing the zero vector is LD.

(Subspaces are LD, but not necessarily the spanning sets)

(6) A set $\{\vec{u}, \vec{v}\}$ is LD if and only if one of the vectors is a multiple of the other one.

(7) A set with three or more vectors can be LD even though no two vectors are multiples of each other.

(8) A set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is LD if and only if there is at least one vector $\vec{v}_k$ which is at least in the span of the rest.