Chapter 6 : The Span of Vectors in a Vector space

All subspaces (except one - $\{0\}$) contain infinite set of vectors but we'll see in this chapter that we can completely identify a subspace by just giving a finite list of vectors.

Example :

(1) Consider $W = \{ \underbrace{(x,y,z) \in \mathbb{R}^3}_{\text{things}} \mid \underbrace{x - 2y + z = 0}_{\text{conditions}} \}$

"conditions" can always be rephrased as one or more equations, eg

$$x - 2y + z = 0 \Rightarrow x = 2y - z,$$

therefore,
$$\begin{aligned} W &= \{ (x,y,z) \in \mathbb{R}^3 \mid x = 2y - z \} \\ &= \{ (2y-z, y, z) \in \mathbb{R}^3 \mid y, z \in \mathbb{R} \} \\ &= \{ (2y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R} \} \\ &= \{ y(2,1,0) + z(-1,0,1) \mid y, z \in \mathbb{R} \} \end{aligned}$$

In other words, W is the "set of all linear combinations" of the vectors $(2,1,0)$ & $(-1,0,1)$. New notation :

$$W = \text{span}\{ (2,1,0), (-1,0,1) \}$$

(W is the span of $(2,1,0)$ & $(-1,0,1)$).

(2) Let's consider $S = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \middle| \ a, b, d \in \mathbb{R} \right\}$, then

$$S = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| \ a, b, d \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Definition of span (section 6.3):

(1) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are vectors in V and $a_1, a_2, \ldots, a_m \in \mathbb{R}$, then

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_m \vec{v}_m$$

is called a "linear combination" of $\vec{v}_1, \ldots, \vec{v}_m$.

(2) The set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$ is called the span of $\vec{v}_1, \ldots, \vec{v}_m$:

$$\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \{ a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_m \vec{v}_m \mid a_1, \ldots, a_m \in \mathbb{R} \}$$

The set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is called the "spanning set".

(3) A vector space is spanned by $\vec{v}_1, \ldots, \vec{v}_m \in V$, if

$$V = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$$

- In the example (1) above, we have: $W = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1 = (2, 1, 0)$ & $\vec{v}_2 = (-1, 0, 1)$.

- In the example (2) above: $S = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.
- Example: $L = \{ t\vec{v} \mid t \in \mathbb{R} \}$ = a line in $\mathbb{R}^n$ through the origin, then

$L = \text{Span}\{\vec{v}\}$, or L is a subspace of $\mathbb{R}^n$ spanned by $\vec{v}$.

Theorem (6.1): (spanned sets are subspaces).

Let V be a vector space. If $\{\vec{v}_1, \dots, \vec{v}_m\} \subset V$, then

(1) $U = \text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ is always a subspace of V.

(2) If W is any subspace of V containing all $\vec{v}_1, \dots, \vec{v}_m$, then $U \subset W$. In fact, U is the smallest subspace that contains $\vec{v}_1, \dots, \vec{v}_m$.

<u>proof of part (1)</u>: subspace test:

- $\vec{0} = 0\vec{v}_1 + \dots + 0\vec{v}_m \in U$
- If $\vec{u} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ and $\vec{w} = b_1\vec{v}_1 + \dots + b_m\vec{v}_m$, then

$$\vec{u} + \vec{w} = (a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \dots + (a_m + b_m)\vec{v}_m,$$

and since this is also a linear combination of $\vec{v}_1, \dots, \vec{v}_m$, it is again in U, or U is closed under addition.

- If $\vec{u} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ and $k \in \mathbb{R}$, then

$$k\vec{u} = ka_1\vec{v}_1 + ka_2\vec{v}_2 + \dots + ka_m\vec{v}_m \in U,$$

i.e. U is closed under scalar multiplication.

Thus, U is a subspace of V.

proof of part 2: Exercise (Hint: use closure properties) $\square$

Example: $W = \{ (x, y, x-y) \mid x, y \in \mathbb{R} \}$

$= \{ x(1,0,1) + y(0,1,-1) \mid x, y \in \mathbb{R} \}$

$= \text{Span}\{ (1,0,1), (0,1,-1) \}$ is a subspace of $\mathbb{R}^3$.

Example: $W = \text{Span}\{ \sin(x), \cos(x) \}$ is a subspace of $V = F(\mathbb{R})$.

Example: The set $W = \{ a(1,0,1) + b(2,1,1) \mid a, b \geq 0 \}$ is NOT a subspace, since it is not the span of vectors $(1,0,1)$ & $(2,1,1)$, because by definition the span is the set of all linear combinations, i.e. a, b must be allowed to take on all real numbers.

Definition (6.2): The "trace" (tr) of an $n \times n$ square matrix A is defined as the sum of its diagonal elements:

e.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boxed{tr(A) = a + d} \in \mathbb{R}$.

Example: show that the set $SL_2 = \{ A \in M_{2\times2}(\mathbb{R}) \mid tr(A) = 0 \}$ is a subspace of $M_{2\times2}(\mathbb{R})$.

$SL_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$.

so,

$SL_2 = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

therefore, SL_2 is a subspace of $M_{2\times2}(\mathbb{R})$.

Example: $D_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R} \right\}$

$= \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, d \in \mathbb{R} \right\}$

$= \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a subspace of $M_{2\times2}(\mathbb{R})$

- All the subspaces of $\mathbb{R}^n$:

(1) $\mathbb{R}$: has only to subspaces: the zero space $\{0\}$ & $\mathbb{R}$.

(2) $\mathbb{R}^2$: The only subspaces of $\mathbb{R}^2$ are

(I) the zero subspace $\{0\}$

(II) lines through the origin,

(III) all of $\mathbb{R}^2$.

proof of (2): suppose W be an arbitrary subspace of $\mathbb{R}^2$:

- If it is not zero subspace, then it contains at least one non-zero vector $\vec{v}$. Since W is a subspace, W contains span $\{\vec{v}\}$.
- But, span $\{\vec{v}\}$ = line through origin with direction vector $\vec{v}$.
- If $W \neq$ span $\{\vec{v}\}$, it must contain a vector like $\vec{w}$ not in that line. Then W must contain span $\{\vec{v}, \vec{w}\}$. so, span $\{\vec{v}, \vec{w}\} \overset{?}{=} \mathbb{R}^2$.
- Last step: proof of span $\{\vec{v}, \vec{w}\} = \mathbb{R}^2$:

Let $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ and (x, y) some arbitrary element of $\mathbb{R}^2$. To show that $(x,y) \in$ span $\{\vec{v}, \vec{w}\}$, we have to show that we can solve the following equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + b\begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

for some $a, b \in \mathbb{R}$, which is the same as solving the linear system of equation for a, b:

$$\begin{cases} a v_1 + b w_1 = x \\ a v_2 + b w_2 = y \end{cases}$$

After solving: $a = \dfrac{x w_2 - y w_1}{v_1 w_2 - v_2 w_1}$, $b = \dfrac{v_1 y - v_2 x}{v_1 w_2 - v_2 w_1}$

(Note that $\vec{v}$ & $\vec{w}$ are not parallel, so the area of the parallelogram they generate $|v_1 w_2 - v_2 w_1| \neq 0$). $\Rightarrow$ $\underline{\text{span}\{\vec{v}, \vec{w}\} = \mathbb{R}^2!}$ □

(if $\vec{u}$ & $\vec{v}$ are not parallel)

(3) All the subspaces of $\mathbb{R}^3$:

The only subspaces of $\mathbb{R}^3$ are:

(I) the zero subspace,

(II) lines through the origin,

(III) planes through the origin,

(IV) all of $\mathbb{R}^3$.

(we will see an algebraic proof later).

*<u>problem (1)</u>: Show that span$\{(0,1,1), (1,0,1)\}$ = span$\{(1,1,2), (-1,1,0)\}$

Solution (a): span of two collinear vectors is a plane through the origin.

Find the normal vectors to ~~that~~ planes for both sets:

$(0,1,1) \times (1,0,1) = (1,1,-1) \Rightarrow$ plane $x+y-z=0$.

$(1,1,2) \times (-1,1,0) = (-2,-2,2) \Rightarrow$ plane $-2x-2y+2z=0$

These equations describe the same plane.

Solution (b): use theorem (6.1):

$$\left.\begin{array}{l}(1,1,2) = 1(0,1,1) + 1(1,0,1) \\ (-1,1,0) = 1(0,1,1) - 1(1,0,1)\end{array}\right\}$$

$(1,1,2), (-1,1,0) \in \text{span}\{(0,1,1), (1,0,1)\}$

Therefore, by the theorem (6.1):

$$\text{span}\{(1,1,2), (-1,1,0)\} \subseteq \text{span}\{(0,1,1), (1,0,1)\}$$

conversely,

$$(0,1,1) = \frac{1}{2}(1,1,2) + \frac{1}{2}(-1,1,0)$$
$$(1,0,1) = \frac{1}{2}(1,1,2) - \frac{1}{2}(-1,1,0)$$

$(0,1,1), (1,0,1) \in \text{span}\{(1,1,2), (-1,1,0)\}$

Therefore, by theorem (6.1):

$$\text{span}\{(0,1,1), (1,0,1)\} \subseteq \text{span}\{(1,1,2), (-1,1,0)\}$$

If we have to sets S_1 and S_2, such that $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$, then $S_1 = S_2$. Hence,

$$\text{span}\{(0,1,1), (1,0,1)\} = \text{span}\{(1,1,2), (-1,1,0)\}$$

□

*It is not easy to tell if two subspaces are equal, based on the spanning sets.

* Problem (2): Show that

$$\text{span}\{(0,1,1),(1,0,1)\} = \text{Span}\{(0,1,1),(1,0,1),(1,1,2),(-1,1,0)\}$$

- Solution (a): From previous problem; all vectors on the right-hand side lie in the plane $x+y-z=0$, so their span cannot be bigger than that, since the span is the "smallest" subspace that contains the given vectors. Also, it cannot be any smaller than the plane, since you can already get any vector on that plane via linear combinations of just two of them.

- Solution (b):

obviously,

$$(0,1,1),(1,0,1) \in \text{Span}\{(0,1,1),(1,0,1),(1,1,2),(-1,1,0)\},$$

Hence,

$$\text{span}\{(0,1,1),(1,0,1)\} \subseteq \text{Span}\{(0,1,1),(1,0,1),(1,1,2),(-1,1,0)\}.$$

on the other hand, from previous example we know that every one of the 4 vectors on the right-hand side lies in the span of $(0,1,1)$ and $(1,0,1)$, so

$$\text{span}\{(0,1,1),(1,0,1),(1,1,2),(-1,1,0)\} \subseteq \text{span}\{(0,1,1),(1,0,1)\}$$

Therefore, the subspaces are equal.

$\square$

Having more vectors in the spanning set <u>does not</u> imply that the subspace they span is larger.