Chapter 5 : Subspaces & spanning sets

Recap : A "vector space" is a set V, which is equipped with two operations (addition & scalar multiplication), such that the algebraic axioms are satisfied. we also introduced a few examples, e.g.

- $\mathbb{R}^n$, $n \geq 1$
- $\mathcal{E}$: a space of linear equations in 3 variables,
- $F[a,b]$: space of functions on the interval $[a,b]$,
- $F[\mathbb{R}]$: space of functions on the real line $\mathbb{R}$,
- $M_{m\times n}(\mathbb{R})$: space of $m\times n$ matrices with real elements.

Recall that a vector space is algebraically indistinguishable from the vector spaces like $\mathbb{R}^n$ (with standard addition & scalar multiplication).

Subspaces of vector spaces :

Definition : A subset W of a vector space V is called a "subspace of V" if it is a vector space with the same operations of addition and scalar multiplication.

Example : Consider $V = \mathbb{R}^2$ (which is a vector space), and let $W = \{(x, 2x) \mid x \in \mathbb{R}\}$ be a subset of V with the same set of operations: addition & scalar multiplication.

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Then, we can check the vector space axioms:

closure : - under addion: $(x, 2x)$ & $(y, 2y) \in W$, then

$$(x, 2x) + (y, 2y) = (\underbrace{x+y}_{z}, 2\underbrace{(x+y)}_{z}) \in W, \quad z \in \mathbb{R}. \checkmark$$

- under scalar multiplication: $(x, 2x) \in W$, $c \in \mathbb{R}$, then

$$c(x, 2x) = (cx, 2(cx)) \in W \quad \checkmark$$

existence : - zero vector $(0, 0) \in W$

- $\vec{0} + \vec{u} = \vec{u}$, for each $\vec{u} \in W$

- If $\vec{u} \in W$, then $-\vec{u} = (-1)\vec{u} \in W$, where $\vec{u} = (x, 2x)$

$\Rightarrow -\vec{u} = (-x, -2x)$ & $(x, 2x) + (-x, -2x) = (0, 0)$

algebraic properties: all the algebraic properties are satisfied, since they hold for any vector in $V = \mathbb{R}^2$.

Therefore, W is a vector space, or $W = \{(x, 2x) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

- <u>Subspace Test (Theorem 5.1)</u>:

If V is a vector space and $W \subset V$, then W is a subspace of V iff:

(1) $\vec{0} \in W$,

(2) W is closed under addition: for every $\vec{u}, \vec{v} \in W$, $\vec{u} + \vec{v} \in W$,

(3) W is closed under scalar multiplication; for every $\vec{u} \in W$, $c \in \mathbb{R}$, $c\vec{u} \in W$

Examples :

(I) Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, e.g.

$$2x - y + \frac{z}{3} = 0, \text{ or}$$

$$P = \{ \vec{u} \in \mathbb{R}^3 \mid \vec{u} \cdot (2, -1, \tfrac{1}{3}) = 0 \}$$

Apply the subspace test :

(1) $\vec{0} \in P$, since $\vec{0} = (0,0,0)$ satisfies $\vec{0} \cdot (2, -1, \frac{1}{3}) = 0$.

(2) closure under addition: suppose $\vec{u}, \vec{v} \in P$, i.e.

$\vec{u} \cdot (2, -1, \frac{1}{3}) = 0$ and $\vec{v} \cdot (2, -1, \frac{1}{3}) = 0$, then we find

$$(\vec{u} + \vec{v}) \cdot (2, -1, \tfrac{1}{3}) = \underbrace{\vec{u} \cdot (2, -1, \tfrac{1}{3})}_{0} + \underbrace{\vec{v} \cdot (2, -1, \tfrac{1}{3})}_{0} = 0$$

so, $\vec{u} + \vec{v} \in P$ too.

(3) closure under scalar multiplication: suppose $\vec{u} \in P$, $c \in \mathbb{R}$, i.e.
$\vec{u} \cdot (2, -1, \frac{1}{3}) = 0$ and we have to check if $c\vec{u} \in P$:

$$(c\vec{u}) \cdot (2, -1, \tfrac{1}{3}) = c\big(\underbrace{\vec{u} \cdot (2, -1, \tfrac{1}{3})}_{0} \big) = 0 \Rightarrow c\vec{u} \in P$$

so, "P" is a subspace of $\mathbb{R}^3$.

<u>Note</u> : Any plane in $\mathbb{R}^3$ that does not go through the origin is <u>NOT</u> a subspace of $\mathbb{R}^3$.

(II) Any line through the origin in $\mathbb{R}^n$ is a subspace, i.e. the line $L = \{t\vec{v} \mid t \in \mathbb{R}\}$ in $\mathbb{R}^n$ with the direction vector $\vec{v}$ is a subspace of $\mathbb{R}^n$.

proof: Apply the subspace test:

(1) $\vec{0} \in L$, since for $t=0$, the zero vector $\vec{0} \in L$.

(2) Closure under addition: $t, s \in \mathbb{R}$, $\vec{v} \in L \Rightarrow t\vec{v}, s\vec{v}$ are two points in L. Then $t\vec{v} + s\vec{v} = (t+s)\vec{v} \in L$ (since $(t+s)\vec{v}$ is again a multiple of $\vec{v}$).

(3) Closure under scalar multiplication: If $t\vec{v} \in L$ and $c \in \mathbb{R}$, then $c(t\vec{v}) = (ct)\vec{v} \in \vec{L}$ (since $(ct)\vec{v}$ is a multiple of $\vec{v}$).

Therefore, L is a "subspace" of $\mathbb{R}^n$.

(III) The set of "polynomial functions", i.e. the set $\mathbb{P}$ consisting of all functions that can be written as

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad (a_i \in \mathbb{R}, n \geq 0)$$

is a "subspace" of the vector space $F[\mathbb{R}]$ of all functions in $\mathbb{R}$.

Apply the subspace test:

(1) $p(x) = 0 + 0x + 0x^2 + \cdots = 0$ is the zero polynomial in $\mathbb{P}$.

(2) Closure under addition: sum of two polynomials is again a polynomial.

(3) Closure under scalar multiplication: multiplying a polynomial by a scalar gives another polynomial.

Definition: The "transpose" of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. For instance

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

(IV) The set of "symmetric" 3x3 matrices, i.e the set

$$S = \{ A \in M_{3\times3}(\mathbb{R}) \mid A^T = A \}.$$

is a vector space (or a subspace of $M_{3\times3}(\mathbb{R})$ with the usual matrix operations).

proof: $S \subset M_{3\times3}(\mathbb{R})$, such that $A^T = A$, then (from above)

$a_2 = b_1$, $a_3 = c_1$, $b_3 = c_2$, therefore we can re-write A as:

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \quad a,b,c,d,e,f \in \mathbb{R} \Rightarrow A \in S$$

Apply the subspace test:

(1) Take $a=0, b=0, c=0, d=0, e=0, f=0 \Rightarrow$ zero matrix $\in S$.

(2) Closure under addition: Let A & $A' \in S$, then

$$A + A' = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} + \begin{pmatrix} a' & b' & c' \\ b' & d' & e' \\ c' & e' & f' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' & c+c' \\ b+b' & d+d' & e+e' \\ c+c' & e+e' & f+f' \end{pmatrix} \in S$$

(3) Closure under scalar multiplication: $A \in S$ & $c \in \mathbb{R}$, then $cA \in S$. Therefore, S is a subspace of $M_{3\times3}(\mathbb{R})$.