Chapter 4 : "vector Spaces" :

Goal: looking for non-geometric, non-$\mathbb{R}^n$ vectors

Example: Spaces of Equations

Consider the following three equations:

$$\left.\begin{array}{l} E_1 : x-y-z=1 \\ E_2 : 2x-y+z=1 \\ E_3 : -x+2y+4z=4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} E_2-2E_1 : y+3z=3 \\ E_1+E_3 : y+3z=3 \end{array}\right\}$$

$$\Rightarrow E_2-2E_2 = E_1+E_3$$

$$\text{(or)} \quad 3E_1-E_2+E_3=0$$

Remarks:

(1) we can "add" equations to get another equation.

(2) We can "multiply" an equation "by a scalar" to get another equation.

(3) There is a "zero" equation given by $0=0$. (Let's call it E_0).

(4) Equations have "negatives".

(5) The usual rules of arithmetic hold.

- These are exactly the properties of vectors in n-space $\mathbb{R}^n$.
- Consider the set of all linear combinations of equations E_1, E_2 and E_3, i.e.

$$\mathcal{E} = \{K_1E_1+K_2E_2+K_3E_3 \mid K_1, K_2, K_3 \in \mathbb{R}\}$$

In fact, $\mathcal{E}$ itself acts like a space of vectors algebraically.

<u>Definition</u> : (Vector space)

A "vector space" consists of the following:

(1) A set <u>V</u> (to whose elements we sometimes refer as <u>vectors</u>).

(2) An "addition" operation for two vectors of V (denoted by +).

(3) A "scalar multiplication" of a vector of V by an element $c \in \mathbb{R}$.

such that the following axioms are satisfied:

(1) If $\vec{u}$ and $\vec{v} \in V$, then $\vec{u}+\vec{v} \in V$.

(2) If $\vec{u} \in V$ and $c \in \mathbb{R}$, then $c\vec{u} \in V$.

Closure axioms.

(3) $\vec{0} \in V$: $\vec{0}+\vec{u} = \vec{u}$, for all $\vec{u} \in V$.

(4) Given $\vec{u} \in V$, there exists another vector $-\vec{u} \in V$, such that $\vec{u}+(-\vec{u})=\vec{0}$

Existence.

(5) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$. for all $\vec{u},\vec{v} \in V$.

(6) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$, for all $\vec{u},\vec{v},\vec{w} \in V$.

(7) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$, for all $\vec{u},\vec{v} \in V$ and $c \in \mathbb{R}$.

(8) $(c+d)\vec{u} = c\vec{u}+d\vec{u}$, for all $\vec{u} \in V$ and $c,d \in \mathbb{R}$.

(9) $c(d\vec{u}) = (cd)\vec{u}$, for all $\vec{u} \in V$ and $c,d \in \mathbb{R}$.

(10) $1\vec{u} = \vec{u}$, for all $\vec{u} \in V$.

· <u>Examples</u> : · $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ are all vector spaces, with the usual operations, i.e. addition of vectors and scalar multiplication.

· The set of all linear equations in <u>n</u>-variables, with the usual operations.

· The "zero vector space" $V=\{0\}$: $0+0=0$, $c.0=0$.

- The set $V = \{(x, 2x) \mid x \in \mathbb{R}\}$, with the standard operations from $\mathbb{R}^2$ is a vector space. (check!).
- The set $V = \{(x, x+2) \mid x \in \mathbb{R}\}$, with the standard operations from $\mathbb{R}^2$ is NOT a vector space. (check!).
- The set of complex numbers $\mathbb{C}$, with the algebra of complex numbers forms a vector space. (check).

Definition: A "matrix" is a table of numbers, usually enclosed in square brackets. Its size is $m \times n$ if it has m rows and n columns.

Remarks:
- matrices of the same size can be added entry-wise.
- we can multiply any matrix by a scalar entry-wise as well.
- when we fix m and n, the set $M_{m\times n}$ of all $m\times n$ matrices form a vector space, for instance:

$$M_{2\times 2}(\mathbb{R}) \equiv M_{22}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}.$$

is a vector space. (check closure, existence and also the arithmatic properties from the list of axioms):

(1) closure: $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix} \in M_{22}(\mathbb{R})$

for all $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$.

$r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \in M_{22}(\mathbb{R})$. for all $r, a, b, c, d \in \mathbb{R}$.

(3) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{22}(\mathbb{R})$

(4) $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in M_{22}(\mathbb{R})$, such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(zero matrix in $M_{22}(\mathbb{R})$).

(5) $\vec{u} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$, $c, d \in \mathbb{R}$, then

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 \\ u_3 + v_3 & u_4 + v_4 \end{pmatrix} = \vec{v} + \vec{u}$$

(6) $$\vec{u} + (\vec{v} + \vec{w}) = \begin{pmatrix} u_1 + (v_1 + w_1) & u_2 + (v_2 + w_2) \\ u_3 + (v_3 + w_3) & u_4 + (v_4 + w_4) \end{pmatrix} = (\vec{u} + \vec{v}) + \vec{w}$$

(7) $$c(\vec{u} + \vec{v}) = c\begin{pmatrix} u_1 + v_1 & u_2 + v_2 \\ u_3 + v_3 & u_4 + v_4 \end{pmatrix} = c\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} + c\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$$

$$= c\vec{u} + \vec{v}$$

(8) $$(c+d)\vec{u} = (c+d)\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} (c+d)u_1 & (c+d)u_2 \\ (c+d)u_3 & (c+d)u_4 \end{pmatrix}$$

$$= c\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} + d\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = c\vec{u} + d\vec{u}$$

(9) $$c(d\vec{u}) = c\begin{pmatrix} du_1 & du_2 \\ du_3 & du_4 \end{pmatrix} = \begin{pmatrix} cdu_1 & cdu_2 \\ cdu_3 & cdu_4 \end{pmatrix} = cd\vec{u}$$

(10) $$1\vec{u} = \begin{pmatrix} 1u_1 & 1u_2 \\ 1u_3 & 1u_4 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \vec{u} \in M_{22}(\mathbb{R})$$

Therefore, $M_{22}(\mathbb{R})$ is a vector space.

• space of functions:

Let $[a,b]$ denote the interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ and $F[a,b]$ the set of all functions "f" with domain $[a,b]$ and with values in $\mathbb{R}$, i.e. $F[a,b] = \{f \mid f: [a,b] \to \mathbb{R}\}$.

For $f, g \in F[a,b]$; $f = g$ if and only if $f(x) = g(x)$, for all $x \in [a,b]$.

• Operations in $F[a,b]$:

(1) $(f+g)(x) = f(x) + g(x)$, for all $x \in [a,b]$,

(2) $(cf)(x) = c(f(x))$, for any $c \in \mathbb{R}$ and $x \in [a,b]$.

with these operations, $F[a,b]$ is a vector space. (check!)

(Hint: zero vector is the zero function, which sends every x to $0 \in \mathbb{R}$.)

Example: $F[\mathbb{R}]$: the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

With the same addition and scalar multiplication defined above, $F[\mathbb{R}]$ is also a vector space.

closure and existence criteria hold, with zero vector as the zero function and negative of a function f as the function $-f$, which sends x to $-f(x)$, for all $x \in \mathbb{R}$.

The algebraic operation axioms all hold as well.

Note: $\frac{1}{x}$, $\tan(x) \notin F[\mathbb{R}]$, because they are not defined over all of $\mathbb{R}$.