

Chapter 1: Complex Numbers

①

Hierarchy of number systems:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ Natural numbers

Cannot solve
 $x+5=1$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ Integers

Cannot solve
 $2x=5$

$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$ Rationals

Cannot solve
 $x^2=2$

$\mathbb{R} = \left\{ \begin{array}{l} \text{rationals and irrationals} \\ \text{(like } \sqrt{2}, e, \pi, \dots) \end{array} \right\}$ Real numbers

Cannot solve
 $x^2=-1$

$\mathbb{C} = \left\{ \text{complex numbers} \right\}$ can solve all of the above!

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Denote one solution of $x^2+1=0$ by "i" (stands for "imaginary")
(Euler)

$$\boxed{i^2 := -1} \quad \text{or} \quad \boxed{i := \sqrt{-1}}$$

For any negative real number "a", define:

$$\boxed{\sqrt{a} := (\sqrt{|a|})i}$$

(2)

Definition : The set of "complex numbers"

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

denote $z = a + bi$, then

$$z = a + bi \in \mathbb{C}, \quad a, b \in \mathbb{R}$$

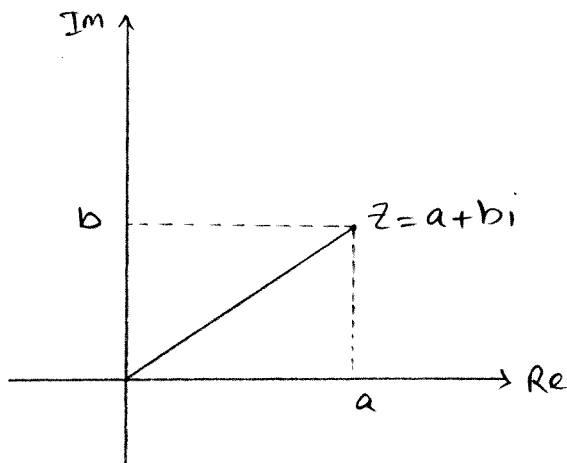
where

$$\begin{cases} a = \operatorname{Re}(z) = \underline{\text{real part of } z} \\ b = \operatorname{Im}(z) = \underline{\text{imaginary part of } z} \end{cases}$$

Example : $b = 0$, then $z = a \in \mathbb{R}$ (a is a real number)

$a = 0$, then $z = bi \in \mathbb{C}$ (called purely imaginary)

Remark : Real numbers form the real number line (or axis);
Complex numbers form the complex "plane":



$$\underline{i^2 = -1}$$

Algebra of Complex Numbers (Section 1.2)

③

- Equality: $a+bi = c+di \Leftrightarrow a=c$ and $b=d$;
- Addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$
- Multiplication: $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$
(Exercise)

- Subtraction: $(a+bi) - (c+di) = (a-c) + (b-d)i$
- Negatives: $-(a+bi) = -a-bi$

Remark: \mathbb{C} satisfies all the usual arithmetical properties of \mathbb{R} , except \mathbb{C} has no "good" order, e.g. " $z > 0$ " makes no sense in general

- Division: (via "rationalizing the denominator")

$$\begin{aligned} \text{First, note } z \cdot \bar{z} &= (a+bi)(a-bi) = a^2 - abi + abi + (bi)(-bi) \\ &= a^2 - b^2 i^2 = a^2 + b^2 = |z|^2 \end{aligned}$$

(Here, $\bar{z} = a-bi$ denotes the "complex conjugate" of $z = a+bi$,
and $|z| = \sqrt{a^2 + b^2}$ denotes the "absolute value", or "modulus", of z)

Now, consider two complex numbers:

$$z = a+bi \in \mathbb{C}$$

$$w = c+di \in \mathbb{C}$$

and the question is to find $\frac{z}{w}$.

$$\begin{aligned} \frac{z}{w} &= \frac{z}{w} \cdot 1 = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd) + (bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i \quad (\text{a complex number!}) \end{aligned}$$

Example : $z = 2+3i$, $w = -3-i$

$$\begin{aligned} \frac{z}{w} &= \frac{2+3i}{-3-i} \cdot \frac{-3+i}{-3+i} = \frac{(2+3i)(-3+i)}{(-3)^2 + (-1)^2} \\ &= \frac{-6+2i-9i+3i^2}{10} = \frac{-9-7i}{10} = -\frac{9}{10} - \frac{7}{10}i \end{aligned}$$

Some other Properties of Complex Numbers : suppose $z, w \in \mathbb{C}$, $a \in \mathbb{R}$

$$1) \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2}$$

$$\Rightarrow \bar{\bar{z}} = z \text{ iff } z \in \mathbb{R}$$

$$2) \overline{z+w} = \bar{z} + \bar{w}$$

$$8) \bar{\bar{z}} = -z \text{ iff } "z" \text{ is purely imaginary}$$

$$9) |z| \in \mathbb{R} \ \& \ |z| \geq 0$$

$$3) \overline{az} = a\bar{z}$$

$$10) |z| = |\bar{z}|$$

$$4) \overline{z\omega} = \bar{z}\bar{\omega}$$

$$11) |z\omega| = |z| \cdot |\omega|$$

$$5) \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$12) \left| \frac{z}{w} \right| = \frac{|z|}{|w|}$$

$$6) \overline{(\bar{z})} = z$$

$$13) |z+w| \leq |z| + |\omega|$$

("Triangle inequality")