Student name and no.

Carleton University

Department of Systems and Computer Engineering

SYSC4505 - Automatic Control Systems – Midterm Exam

October 30, 2014 11:30am-1:00pm

Problem #1 (20%)

a) **(10%)** Given the Laplace transforms $\boldsymbol{F}(\boldsymbol{s}) = \frac{\boldsymbol{s}^2 + 2\boldsymbol{s} + 3}{(\boldsymbol{s} \pm 1)^3}$, find *f(t)* for both cases.

b) **(10%)** By using the Laplace Transform final value theorem find $\lim_{t \to \infty} \boldsymbol{f}(\boldsymbol{t})$ and compare with your findings from part a).

Problem 1 (solution)

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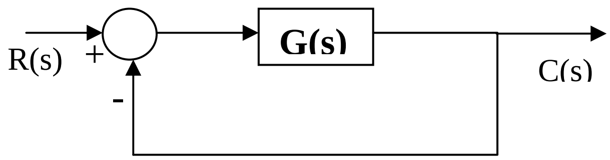
Problem #2 (80%)

Consider the unity feedback system below.



Please note that parts a)-e) can be solved independently of each other.

a) **(15%)** For $G(s)=\dfrac{K}{s(s^2+9s+18)}$ and K>0. Determine the range of K so that the system is stable. For what value of K the system becomes oscillatory and what is the frequency of the oscillation?

b) (**30%**) Construct an approximate plot of the root locus for K>0 and provide a brief explanation of your approach for the construction.

c) **(10%)** Assume that you have an accurate root locus plot generated by using a computer software such as MATLAB. Provide a graphical procedure for determining the value of K that will result in a system having two complex conjugate poles with a damping factor $\xi=0.5$. No calculations are needed just a brief description/sketch that explain the methodology.

d) **(15%)** Assuming that in part c) you determined that the two complex closed loop poles corresponding to a damping factor $\xi=0.5$ are $s_{1,2}=-1\pm j1.732$. Determine the value of K that will result in the system having these two poles. Are these two poles dominant? Explain.

e) **(10%)** For K=28 find the steady state errors of the system due to a step and due to a ramp input.

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Problem 2 (solution)

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Problem 2 (solution)

SYSC4505 - Midterm Exam Solution - Fall 2014.

a) $F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$

For details please Check textbook on p. 869-870-871.

$b_1 = F(s)(s+1)^3 \Big|_{s=-1} = 2$

$b_2 = \frac{d}{ds}\left[F(s)(s+1)^3\right]\Big|_{s=-1} = \quad (2s+2)\Big|_{s=-1} = 0$

$b_3 = \frac{1}{2}\frac{d^2}{ds^2}\left[F(s)(s+1)^3\right]\Big|_{s=-1} = \frac{1}{2}\frac{d}{ds}(2s+2)\Big|_{s=-1}$

$= 1.$

Hence $F(s) = \frac{1}{s+1} + \frac{2}{(s+1)^3} \xrightarrow{\mathcal{L}^{-1}} F(t) = e^{-t} + t^2 e^{-t}$

For $F(s) = \frac{s^2+2s+3}{(s-1)^3} = \frac{b_1}{s-1} + \frac{b_2}{(s-1)^2} + \frac{b_3}{(s-1)^3}$

$b_3 = F(s)(s-1)^3\Big|_{s=1} = 6.$

$b_2 = \frac{d}{ds}\left(F(s)(s-1)^3\right)\Big|_{s=1} = (2s+2)\Big|_{s=1} = 4.$

$b_3 = \frac{1}{2}\frac{d^2}{ds^2}\left(F(s)(s-1)^3\right)\Big|_{s=1} = 1$

Hence $F(s) = \frac{6}{s-1} + \frac{4}{(s-1)^2} + \frac{1}{(s-1)^3}$

Therefore:

$$F(t) = 6e^t + 4te^t + 3t^2e^t$$

b) For $F(t) = \frac{s^2+2s+3}{(s+1)^3}$ $\lim_{t\to\infty} f(t) = \lim_{t\to 0} sF(s) =$

$$= \lim_{s\to 0} s \cdot \frac{s^2+2s+3}{(s+1)^3} = \underline{\underline{0}}.$$

As expected $\lim_{t\to\infty} (e^{-t} + t^2e^{-t}) =$

$$= \lim_{t\to\infty} e^{-t} + \lim_{t\to\infty} t^2e^{-t} = 0 + \lim_{t\to\infty} \frac{t^2}{e^t} =$$

$$= \lim_{t\to\infty} \frac{(t^2)'}{(e^t)'} = \lim_{t\to 0} \frac{2t}{e^t} = \lim_{t\to\infty} \frac{2}{e^t} = \underline{\underline{0}}.$$

↑ Hospital's Rule

$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$ is valid here because $F(t)$ has poles in the Left Half plane.

This is not the case for $F(t) = \frac{s^2+2s+3}{(s-1)^3}$!

$\lim_{s\to 0} sF(s) = 0$ similar as before but this is not valid since as it can be seen

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} (6e^t + 4te^t + 3t^2e^t) = +\infty!!!$$

Problem #2 This problem is virtualy identical (same GH) as example A-6-19 in the book, page 390.

a) The characteristic equation is:

$$1 + KG(s)H(s) = 1 + k\,G(s) = 0 \Rightarrow$$

$$s^3 + 9s^2 + 18s + k = 0.$$

Routh-Hurwitz table:

s^3	1	18
s^2	9	k
s^1	$\frac{162-k}{9}$	0
s^0	k	0

Hence $k > 0$ and $162 - k > 0 \Rightarrow k < 162$

Therefore system is stable for $0 < k < 162$

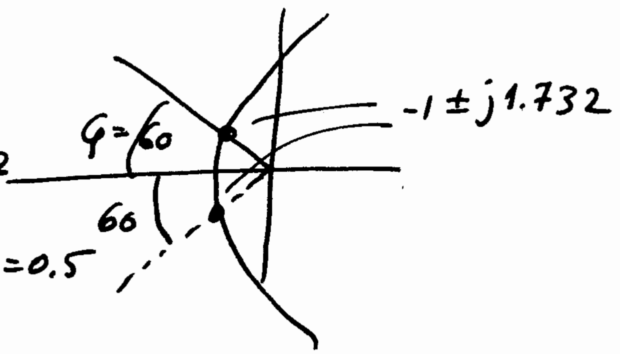
For $K = 162$ the system will become oscilatory i.e it will have closed loop poles on the imaginary axis. These can be found for $K = 162$ from the aux. polynomial. $9s^2 + 162 = 0 \Rightarrow s^2 = 18$

$$\Rightarrow s_{1,2} = \pm j3\sqrt{2}$$

Hence the frequeny of oscillation is $3\sqrt{2}$ radi/sec. in this case.

b) The root locus for this problem along with comments etc can be found in page 390-391, fig. 6-98 of the book

c) If we have the detailed root locus as in fig 6-98 then we draw a line at an angle φ of 60° as shown in the figure. The two complex conjugate poles are then $s_{1,2} = -1 \pm j1.732$



These poles will have a $\xi = 0.5$

Also $|k\,G(s_{1,2})| = 1 \Rightarrow$ (magnitude condition)

$$k\,|G(-1+j1.732)| = 1 \Rightarrow$$

$$\Rightarrow k = \frac{1}{|G(-1+j1.732)|}.$$

$$G(s) = \frac{1}{s(s^2+9s+18)} = \frac{1}{s(s+3)(s+6)} \Rightarrow$$

$$\Rightarrow \frac{1}{|G(-1+j1.732)|} = \frac{1}{|(-1+j1.732)(2+j1.732)(7+j1.732)|}$$

mag = 2   mag = 2.64   mag = 7.2

$= \frac{1}{\ldots}$

$$\frac{1}{|G(-1+j1.732)|} = \left|(-1+j1.732)(2+j1.732)(5+j1.732)\right|$$

$\uparrow$ mag = 2 $\quad\uparrow$ mag = 2.64 $\quad\uparrow$ mag. = 5.29

$= 28 \Rightarrow \underline{\underline{k = 28}}$

For $k=28$ the sum of the poles is -9

$$\underbrace{s_1 + s_2}_{-2} + s_3 = -9 \Rightarrow s_3 = -7.$$

$|-7|$ is 5 times (in fact 7 times) larger than $|Re(-1 \pm j1.732)| = 1$

hence $-1 \pm j1.732$ are dominant and the system can be approximated by a 2nd order system with these poles.

d) Since $GH = G(s) = \dfrac{28}{s(s^2+9s+18)}$

the system is type-1 and the step input error is 0. For the ramp

$$e_{ss} = \underbrace{\lim_{s\to 0} \frac{1}{sG(s)}}_{\text{(why?)}} = \lim_{s\to 0} \frac{1}{\frac{28}{s^2+9s+18}} = \frac{18}{28}.$$