1. (20 marks) Consider a two period, 0 and 1, and Ω states of the world economy in which there is a risky asset x available at time 0 and its payoffs at time 1 are also denoted by x. Suppose one agent with utility function $U(c_0, c_1)$ has consumption good endowment $(w_0, w_1) \in \mathbb{R}_{++}^{1+\Omega}$ and y units of asset endowment, respectively.

   (a) (6 marks) Let $y = 1$. Suppose that the agent is going to sell the asset at time 0. What is the minimum price CE(x) he is going to ask?

   (b) (7 marks) Let $y = 0$. Suppose that the agent is going to buy one unite of it at time 0. What is the maximum price $P(x)$ he is going to bid?

   (c) (7 marks) Let $y = 1$. Suppose that the agent is going to buy insurance for protecting the loss at some situations at time 0. What is the maximum risk premium $\rho(x)$ he is going to pay?

   (**Hint: For each of the questions, just specify the equation that the price should satisfy.**)

2. (20 marks) Consider an economy with two periods, 0, 1, and one state of the world at time 1 (i.e. there is no uncertainty in this economy) and one perishable consumption good. Agent A has endowment $(100, 1)$ and utility function

$$U(c_0, c_1) = \ln c_0 + 0.5 \ln c_1.$$

   There is one financial asset in the economy with payoff $1 + r_f$ at time 1 and price 1 at time 0, where $r_f > 0$ is the interest rate.

   (a) (5 marks) Suppose that there is **no market** for the financial asset at time 0. Find his optimal consumption bundle and his utility at this consumption bundle. Denote his utility U^{no} for this case.

   (b) (10 marks) Now, suppose that there is a market for the financial asset at time 0. Find his optimal consumption bundle and his optimal portfolio. Denote his utility U^{yes} for this case.

   (c) (5 marks) Is $U^{yes} > U^{no}$? Why or why not? What can you conclude about the function of the financial market?

3. (20 marks) Consider an economy consisting of (i) two periods and two states of the nature at time 1 with the same probability $\frac{1}{2}$ and (ii) I (≥ 2) strictly monotone and strictly risk averse agents with the total endowment

$$e = \sum_{i=1}^{I} e^i = (e_0, e_1) = (e_0, e_{11}, e_{12}) = (10, 100, 20).$$

   There is two assets in the economy as follows:

x^1 • ↗ 25 if state 1; ↘ 5 if state 2

   and

x^2 • ↗ 5 if state 1; ↘ 25 if state 2

   (a) (10 marks) Calculate $\mathbb{E}(x^1)$, $\mathbb{E}(x^2)$, $\text{cov}(e_1, x^1)$, and $\text{cov}(e_1, x^2)$, respectively.

   (b) (10 marks) Suppose there is a representative agent in the economy. Is it possible that the equilibrium prices of the two assets are the same? Why or why not? State your reasons clearly. (Hint: Try to use CCAPM (Consumption Based Capital Asset Pricing Model) to explain this.)

4. (20 marks) Given the price-payoff couple (S, X) as follows:

$$S = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1.5 \\ 0.5 \\ 0.25 \end{pmatrix}$$

and

$$X = \begin{pmatrix} x^1 & x^2 & x^3 & x^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

(a) (2 marks) Are the markets complete?

(b) (3 marks) Can the Arrow-Debreu security AD^2 be replicated? If yes, find a portfolio θ such that

$$X\theta = AD^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) (5 marks) Do there exist arbitrage opportunities in the markets? Why or why not? State your reasons clearly.

(d) (10 marks) If there are arbitrage opportunities in the markets, is it possible to change asset 4's price **only** such that there are no more arbitrage opportunities? Why or why not?

5. (20 marks)

(a) (5 marks) Write the Arrow-Pratt absolute and relative risk aversion coefficients $A(x)$ and $R(x)$, respectively

(b) (5 marks) State the two concepts of absolute and relative risk, respectively.

(c) (10 marks) Consider two persons, a and b, with utility indexes

$$u_a(x) = \frac{1}{1-b} x^{1-b} \ (x > 0, b \geq 0 \text{ and } \neq 1)$$

and

$$u_b(x) = \ln x, \quad (x > 0).$$

Are they DARA (decreasing absolutely risk averse) and CRRA (constant relatively risk averse)? Why or why not? State your reasons clearly.

Solution to the final examination

1. (a) The agent has two options at time 0: Keeping the asset and selling it. The price CE(x) should make the agent to be indifferent with the two options. That is, the price CE(x) is a solution to

$$U\left(w_0, w_1 + x\right) = U\left(w_0 + \mathrm{CE}\left(x\right), w_1\right).$$

(b) The agent has two options at time 0: Buying the asset and not buying it. The price $P\left(x\right)$ should make the agent to be indifferent with the two options. That is, the price $P\left(x\right)$ is a solution to

$$U\left(w_0, w_1\right) = U\left(w_0 - P\left(x\right), w_1 + x\right).$$

(c) The agent has two options: Buying the insurance and not buying it at time 0. The risk premium $\rho\left(x\right)$ should make the agent to be indifferent with the two options. That is, the price $\rho\left(x\right)$ is a solution to

$$U\left(w_0, w_1 + x\right) = U\left(w_0 - \rho\left(x\right), w_1 + \mathbb{E}\left(x\right)\right).$$

2. (a) Since there is **no market** for the financial asset, his optimal consumption bundle is his endowment

$$c = e = \left(c_0, c_1\right) = \left(100, 1\right)$$

and his utility at this consumption bundle is

$$U\left(e_0, e_1\right) = \ln 100 + 0.5 \ln 1 = \ln 100.$$

(b) Since there is **a market** for the financial asset with the interest rate $r_f > 0$, the agent's total income at time 0 becomes

$$I = 100 + \frac{1}{1 + r_f} = \frac{101 + 100 r_f}{1 + r_f}$$

and his budget set is

$$B\left(e, r_f\right) = \left\{\left(c_0, c_1\right) \in R_+^{1+1} : c_0 + \frac{c_1}{1 + r_f} \leq \frac{101 + 100 r_f}{1 + r_f}\right\}.$$

The agent solves

$$\max_{\left(c_0, c_1\right) \in B\left(e, r_f\right)} U\left(c_0, c_1\right) = \ln c_0 + 0.5 \ln c_{11}.$$

Since there is no corner solution, we just need to solve the Lagrangean problem as follows:

$$\begin{aligned} \mathcal{L} &= U\left(c_0, c_1\right) - \lambda\left(c_0 + \frac{c_1}{1 + r_f} - I\right) \\ &= \ln c_0 + 0.5 \ln c_1 - \lambda\left(c_0 + \frac{c_1}{1 + r_f} - I\right). \end{aligned}$$

The first-order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_0} &= \frac{1}{c_0} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial c_1} &= \frac{0.5}{c_1} - \lambda \frac{1}{1 + r_f} = 0 \end{aligned}$$

and

$$c_0 + \frac{c_1}{1 + r_f} = I. \tag{1}$$

Accordingly,

$$c_0 = \frac{1}{\lambda}, \; c_1 = \frac{0.5\left(1 + r_f\right)}{\lambda}. \tag{2}$$

Substituting (2) into (1) yields

$$\begin{aligned}\frac{1}{\lambda}+\frac{0.5\left(1+r_f\right)}{\lambda\left(1+r_f\right)} &= \frac{1.5}{\lambda}=I \\ \lambda &= \frac{1.5}{I}.\end{aligned}$$

Thus,

$$\begin{aligned}c_0 &= \frac{1}{\lambda}=\frac{I}{1.5}=\frac{\frac{101+100r_f}{1+r_f}}{1.5}=\frac{101+100r_f}{1.5\left(1+r_f\right)} \\ c_1 &= \frac{0.5\left(1+r_f\right)}{\lambda}=\frac{0.5\left(1+r_f\right)\frac{101+100r_f}{1+r_f}}{1.5}=\frac{101+100r_f}{3}.\end{aligned}$$

The agent's portfolio is

$$\begin{aligned}\theta &= \frac{e_0-c_0}{1}=100-\frac{I}{1.5}=100-\frac{\frac{101+100r_f}{1+r_f}}{1.5} \\ &= 100-\frac{101+100r_f}{1.5\left(1+r_f\right)}=\frac{100\left(1+r_f\right)-1}{3\left(1+r_f\right)}.\end{aligned}$$

(c) $U^{yes}>U^{no}$ since his endowment is within his feasible budget set. The financial market increases agent A's utility.

3. (a)

$$\mathbb{E}x^1=\mathbb{E}x^2=\frac{25+5}{2}=15.$$

Since

$$\begin{aligned}\operatorname{cov}\left(e_1,x^i\right) &= \mathbb{E}\left(e_1-\mathbb{E}e_1\right)\left(x^i-\mathbb{E}x^i\right) \\ &= \mathbb{E}\left(e_1x^i\right)-\mathbb{E}e_1\mathbb{E}x^i,\ i=1,2,\end{aligned}$$

we calculate $\mathbb{E}\left(e_1x^i\right)$ and $\mathbb{E}e_1$.

$$\begin{aligned}\mathbb{E}e_1 &= \frac{100+20}{2}=60 \\ \mathbb{E}\left(e_1x^1\right) &= \frac{100\times25+20\times5}{2}=1300 \\ \mathbb{E}\left(e_1x^2\right) &= \frac{100\times5+20\times25}{2}=500 \\ \mathbb{E}e_1\mathbb{E}x^1 &= \mathbb{E}e_1\mathbb{E}x^2=60\times15=900\end{aligned}$$

$$\begin{aligned}\operatorname{cov}\left(e_1,x^1\right) &= \mathbb{E}\left(e_1-\mathbb{E}e_1\right)\left(x^1-\mathbb{E}x^1\right)=\mathbb{E}\left(e_1x^1\right)-\mathbb{E}e_1\mathbb{E}x^1 \\ &= 1300-900=400>0 \\ \operatorname{cov}\left(e_1,x^2\right) &= \mathbb{E}\left(e_1-\mathbb{E}e_1\right)\left(x^2-\mathbb{E}x^2\right)=\mathbb{E}\left(e_1x^2\right)-\mathbb{E}e_1\mathbb{E}x^2 \\ &= 500-900=-400<0.\end{aligned}$$

(b) No. S^2 should be bigger than S^1. When tomorrow (state 1) is booming, you will have more income from your other resource, and will have lower marginal utility. That is, in booming period, you don't "really" need more money. However, asset x^1 still gives you more money when you don't "really" need it. When tomorrow is in recession, you will have less income from your other resource, and will have higher marginal utility. That is, in recession period, you do "really" need more money. However, asset x^1 does not give you more money when you do "really" need it.

That is, when you do not "really" need more money, asset x^1 gives you more money; However, when you "really" need more money, asset x^1 does not give you more money. Somehow, owning asset x^1 is equivalent to buying insurance against the booming. However, we don't need such insurance. In other words, asset x^1 is a "bad" asset. Therefore, you would not like to pay more for it.

We can we use CCAPM to explain this as following: Asset x^1 and the underlying economy have positive covariance

$$\operatorname{cov}\left(e_1, x^1\right) > 0 \Longrightarrow \operatorname{cov}\left(\frac{u_1'\left(e_1\right)}{u_0'\left(e_0\right)}, x^1\right) < 0$$

(Can you imagine why? Since u_1' is strictly decreasing from $u_1'' < 0$.) As a result,

$$\begin{aligned} &\operatorname{cov}\left(\tfrac{u_1'(e_1)}{u_0'(e_0)}, x^1\right) < 0 \\ &S_1 = \tfrac{\mathbb{E}\left[x^1\right]}{1+r_f} + \operatorname{cov}\left(\tfrac{u_1'(e_1)}{u_0'(e_0)}, x^1\right) < \tfrac{\mathbb{E}\left[x^1\right]}{1+r_f}. \end{aligned}$$

That is, for such a "bad" asset x^1, the risk premium is negative.

How about asset x^2?

When tomorrow is booming, you will have more income from your other resource, and will have lower marginal utility. That is, in booming period, you don't "really" need more money. And asset x^2 also does not give you more money when you don't "really" need it. When tomorrow is in recession, you will have less income from your other resource, and will have higher marginal utility. That is, in recession period, you do "really" need more money. Fortunately, asset x^2 will give you more money when you do "really" need it.

That is, when you do not need more money, asset x^2 also does not give you more money; Fortunately, when you do need more money, asset x^2 also gives you more money. Somehow, owning asset x^2 is equivalent to buying insurance against the recession. In other words, asset x^2 is a "good" asset. Therefore, you would like to pay more for it.

Similarly, we can use CCAPM to explain this. Asset x^2 and the underlying economy have negative covariance

$$\operatorname{cov}\left(e_1, x^2\right) < 0 \Longrightarrow \operatorname{cov}\left(\frac{u_1'\left(e_1\right)}{u_0'\left(e_0\right)}, x^2\right) > 0$$

As a result,

$$\begin{aligned} &\operatorname{cov}\left(\tfrac{u_1'(e_1)}{u_0'(e_0)}, x^2\right) > 0 \\ &S_2 = \tfrac{\mathbb{E}\left[x^2\right]}{1+r_f} + \operatorname{cov}\left(\tfrac{u_1'(e_1)}{u_0'(e_0)}, x^2\right) > \tfrac{\mathbb{E}\left[x^2\right]}{1+r_f}. \end{aligned}$$

That is, for such a "good" asset x^2, the risk premium is positive.

Thus,

$$S_2 > \frac{\mathbb{E}\left[x^2\right]}{1+r_f} = \frac{\mathbb{E}\left[x^1\right]}{1+r_f} > S_1.$$

4. (a) Yes, the markets are complete since

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} = 1,$$

and the rank of X is 4. Thus, the maximum number of linearly independent column vectors is equal to the number of the number of the states of nature. Accordingly, the markets are complete.

(b) Yes since the markets are complete. To find θ, we solve

$$X\theta = AD^2.$$

That is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \theta_1 &= 0 \Longrightarrow \theta_1 = 0 \\ 2\theta_1 + \theta_2 &= 1 \Longrightarrow \theta_2 = 1 - 2\theta_1 = 1 \\ 3\theta_1 + 2\theta_2 + \theta_3 &= 0 \Longrightarrow \theta_3 = -3\theta_1 - 2\theta_2 = -2 \\ 4\theta_1 + 3\theta_2 + 2\theta_3 + \theta_4 &= 0 \Longrightarrow \theta_4 = -4\theta_1 - 3\theta_2 - 2\theta_3 = -3 + 4 = 1. \end{aligned}$$

(c) By the definition, we have

$$S_i = x_1^i\phi_1 + x_2^i\phi_2 + x_3^i\phi_3 + x_4^i\phi_4, \ \ i = 1, 2, 3, 4.$$

Thus,

$$\begin{aligned} S_1 &= x_1^1\phi_1 + x_2^1\phi_2 + x_3^1\phi_3 + x_4^1\phi_4 = \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 \\ S_2 &= x_1^2\phi_1 + x_2^2\phi_2 + x_3^2\phi_3 + x_4^2\phi_4 = 0 + 1\phi_2 + 2\phi_3 + 3\phi_4 \\ S_3 &= x_1^3\phi_1 + x_2^3\phi_2 + x_3^3\phi_3 + x_4^3\phi_4 = 0 + 0 + 1\phi_3 + 2\phi_4 \\ S_4 &= x_1^4\phi_1 + x_2^4\phi_2 + x_3^4\phi_3 + x_4^4\phi_4 = 0 + 0 + 0 + 1\phi_4. \end{aligned}$$

By using matrix form, ϕ is a solution to

$$X^T\phi = S.$$

$$X^T\phi = S.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}^T \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1.5 \\ 0.50 \\ 0.25 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1.5 \\ 0.5 \\ 0.25 \end{pmatrix}.$$

Since the first matrix of the right hand side is an upper triangle matrix, we solve them from the bottom

$$\begin{aligned} \phi_4 &= 0.25 \Longrightarrow \phi_4 = 0.25 \\ \phi_3 + 2\phi_4 &= 0.5 \Longrightarrow \phi_3 = 0.5 - 2\phi_4 = 0 \\ \phi_2 + 2\phi_3 + 3\phi_4 &= 1.5 \Longrightarrow \phi_2 = 1.5 - 2\phi_3 - 3\phi_4 = 1.5 - 0 - 0.75 = 0.75 \\ \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 &= 2.5 \Longrightarrow \phi_1 = 2.5 - 2\phi_2 - 3\phi_3 - 4\phi_4 = 2.5 - 1.5 - 0 - 1 = 0. \end{aligned}$$

Though

$$\phi \geq 0,$$

however,

$$\phi \not\gg 0,$$

there are arbitrage opportunities in the security markets.

(d) It is impossible. From

$$X^T \phi = S,$$

we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}^T \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1.5 \\ 0.50 \\ S_4 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 1.5 \\ 0.50 \\ 0.15 \end{pmatrix}.$$

Since the first matrix of the right hand side is an upper triangle matrix, we solve them from the bottom

$$\begin{aligned} \phi_4 &= S_4 && (3) \\ \phi_3 + 2\phi_4 &= 0.5 && (4) \\ \phi_2 + 2\phi_3 + 3\phi_4 &= 1.5 && (5) \\ \phi_1 + 2\phi_2 + 3\phi_3 + 4\phi_4 &= 2.5 && (6) \end{aligned}$$

Equation 6− equation 5 gives

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 1$$

and Equation 5− equation 4 gives

$$\phi_2 + \phi_3 + \phi_4 = 1$$

Accordingly,

$$\phi_1 = 0.$$

That, there is arbitrage opportunity in the economy!

$$\phi \gg 0,$$

there are no arbitrage opportunities in the security markets.

5. (a) Let utility index u be twice differentiable,

$$A(x) = -\frac{u''(x)}{u'(x)}$$

and

$$R(x) = -\frac{xu''(x)}{u'(x)}.$$

(b) Say risk x to be absolute if it has the following form

$$x = \bar{x} + \epsilon,$$

where $\bar{x}$ is a constant, ϵ is a random variable with

$$E(\epsilon) = 0 \text{ and } \operatorname{var}(\epsilon) = \sigma^2 > 0.$$

Say risk x to be relative if it has the following form

$$x = \bar{x}(1 + \epsilon),$$

where $\bar{x}$ is a constant, ϵ is a random variable with

$$E(\epsilon) = 0 \text{ and } \operatorname{var}(\epsilon) = \sigma^2 > 0.$$

(c) For person a with utility index

$$u_a(x) = \frac{1}{1-b} x^{1-b} \ (x > 0, b \geq 0 \text{ and } \neq 1),$$

we have

$$\begin{aligned} u'(x) &= x^{-b},\ u''(x) = -bx^{-b-1} \\ A(x) &= -\frac{u''(x)}{u'(x)} = \frac{bx^{-b-1}}{x^{-b}} = \frac{b}{x} \\ R(x) &= -\frac{xu''(x)}{u'(x)} = \frac{xbx^{-b-1}}{x^{-b}} = b. \end{aligned}$$

Since

$$\frac{dA(x)}{dx} = -\frac{b}{x^2} < 0$$

and

$$\frac{dR(x)}{dx} = 0,$$

therefore, person a is DARA and CARA.

For person b with utility index

$$u(x) = \ln x, \quad (x > 0),$$

we have

$$u'(x) = \frac{1}{x},\ u''(x) = -\frac{1}{x^2}$$

$$\begin{aligned} A(x) &= -\frac{u''(x)}{u'(x)} = \frac{\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x} \\ R(x) &= -\frac{xu''(x)}{u'(x)} = \frac{x\frac{1}{x^2}}{\frac{1}{x}} = 1. \end{aligned}$$

Since

$$\frac{dA(x)}{dx} = -\frac{1}{x^2} < 0$$

and

$$\frac{dR(x)}{dx} = 0,$$

therefore, person b is DARA and CARA.