Phys 197 Homework Solution 36

Q7. In single-slit diffraction, what is $\sin(\beta/2)$ when $\theta = 0$? In view of your answer, why is the single-slit intensity not equal to zero at the center?

When $\theta = 0$, $\beta = 2\pi a \sin\theta/\lambda = 0$. Physically, the central point is a maximum because all the contributions arrive in phase. Mathematically, the formula has $\beta/2$ in the denominator, so that the limit of the quotient is non-zero.

Q8. A rainbow ordinarily shows a range of colors (see Section 33.4). But if the water droplets that form the rainbow are small enough, the rainbow will appear white. Explain why, using diffraction ideas. How small do you think the raindrops would have to be for this to occur?

Light diffracts. If the angular width of the central maximum for any particular color is about the angular width of the rainbow, the "fogbow" will result. The width of a rainbow is about $2°$, so we need to estimate a value for a that makes $\theta = 2°$. See Eqs 36.5 and 36.6 for θ and β. The size of a droplet would be roughly what it takes to make $\beta/2 = \pi$, or

$\beta = 2\pi = 2\pi \sin\theta/\lambda \quad \Rightarrow \quad a = \lambda/\sin\theta = (500 \text{ nm})/0.035 = 14\ \mu\text{m}.$

This wouldn't be exact, since a factor of 1.22 would probably belong. The estimate of how much smearing is necessary to suppress the colors is also rough.

Q9. Some loudspeaker horns for outdoor concerts (at which the entire audience is seated on the ground) are wider vertically than horizontally. Use diffraction ideas to explain why this is more efficient at spreading the sound uniformly over the audience than either a square speaker horn or a horn that is wider horizontally than vertically. Would this still be the case if the audience were seated at different elevations, as in an amphitheater? Why or why not?

Making them wide vertically means the sound is concentrated in the vertical direction. Making them narrow horizontally distributes the sound in a wide horizontal maximum. For an amphitheater, a wider vertical distribution would be desired.

P3. Light of wavelength 585 nm falls on a slit 0.0666 mm wide. (a) On a very large and distant screen, how many totally dark fringes (indicating complete cancellation) will there be, including both sides of the central bright spot? Solve this problem without calculating all the angles! (Hint: What is the largest that $\sin\theta$can be? What does this tell you is the largest that can be?) (b) At what angle will the dark fringe that is most distant from the central bright fringe occur?

The condition for a dark fringe is $\sin\theta = m\lambda/a$. Since $\sin\theta < 1$ for a screen, we have

$m\lambda/a < 1 \quad \Rightarrow \quad m < a/\lambda = 113.86.$

(a) **226 dark fringes.** They occur for $m = \pm 1,\ \pm 2 ..., \pm 113$.

(b) $\sin\theta = 113\lambda/a = 113 \cdot (0.585\ \mu\text{m})/(66.6\ \mu\text{m}) = 0.99257$

$\theta = \text{Sin}^{-1}(0.99257) = \pm\mathbf{83.0}°.$

P11. Parallel rays of light with wavelength 620 nm pass through a slit covering a lens with a focal length of 40.0 cm. The diffraction pattern is observed in the focal plane of the lens, and the distance from the center of the central maximum to the first minimum is 36.5 cm. What is the width of the slit? (Note: The angle that locates the first minimum is *not* small.)

The lens makes the rays parallel, so doesn't really figure in the calculation. This is $m = 1$, so we have

$\sin\theta = 36.5/\sqrt{36.5^2 + 40^2} = 0.67405$. Plug

$\sin\theta = \lambda/a \quad \Rightarrow \quad a = \lambda/\sin\theta = (0.620\ \mu\text{m})/0.67405 = \mathbf{0.920\,\mu m}$.

P15. A slit 0.240 mm wide is illuminated by parallel light rays of wavelength 540 nm. The diffraction pattern is observed on a screen that is 3.00 m from the slit. The intensity at the center of the central maximum ($\theta = 0$) is 6.00×10^{-6} W/m^2. (a) What is the distance on the screen from the center of the central maximum to the first minimum? (b) What is the intensity at a point on the screen midway between the center of the central maximum and the first minimum?

(a) $\sin\theta = \lambda/a = (0.540\ \mu\text{m})/(240\ \mu\text{m}) = 0.00225 \quad \Rightarrow \quad \theta = 0.1289°$.

$x = R\tan\theta = (3\text{ m})(0.00225) = \mathbf{6.750\,mm}$. The small-angle approximation would have been fine.

(b) Using the small-angle approximation, θ here will be half the previous value, or $\theta = 0.0645°$ and $\sin\theta = 1.126 \times 10^{-3}$ and (see Eq 36.6 p. 1196)

$\beta = 2\pi a\sin\theta/\lambda = 6.2832(240\ \mu\text{m})(1.126 \times 10^{-3})/(0.540\ \mu\text{m}) = 3.144$. So, $\beta/2 = 1.572$. Plug into Eq. 36.5:

$I = I_0\,(\sin(\beta/2)/(\beta/2))^2 = (6.00 \times 10^{-6}\text{ W/m}^2)\,(1.00/1.572)^2 = \mathbf{2.43 \times 10^{-6}\,W/m^2}$.

P21. Number of Fringes in a Diffraction Maximum. In Fig. 36.12c the central diffraction maximum contains exactly seven interference fringes, and in this case $d/a = 4$. (a) What must the ratio be if the central maximum contains exactly five fringes? (b) In the case considered in part (a), how many fringes are contained within the first diffraction maximum on one side of the central maximum?

(a) To get 5 fringes, we have the central max, then two fringes on each side. We will have the zero of the diffraction pattern suppress the third fringe. Then

$d\sin\theta = 3\lambda$ and $a\sin\theta = \lambda$, Dividing,

$d/a = \mathbf{3}$.

(b) Two complete fringes between the central max and the first zero of the diffraction pattern. The center of fringe zero is dead on the center of the diffraction maximum.

P28. Monochromatic light is at normal incidence on a plane transmission grating. The first-order maximum in the interference pattern is at an angle of 8.94°. What is the angular position of the fourth-order maximum?

Maxima occur at $d\sin\theta = m\lambda$, or $\sin\theta = m\lambda/d$. For $m = 1$, $\theta_1 = 8.94° \quad \Rightarrow \quad \sin\theta_1 = 0.1554$.

$\sin\theta_4 = 4\sin\theta_1 = 0.6216 \quad \Rightarrow$

$\theta_4 = \mathbf{38.4}°$.

P36. Identifying Isotopes by Spectra. Different isotopes of the same element emit light at slightly different wavelengths. A wavelength in the emission spectrum of a hydrogen atom is 656.45 nm; for deuterium, the corresponding wavelength is 656.27 nm. (a) What minimum number of slits is required to resolve these two wavelengths in second order? (b) If the grating has 500.00 slits/mm, find the angles and angular separation of these two wavelengths in the second order.

(a) Resolving Power is defined as $R = \lambda/\Delta\lambda$, and is given by $R = mN$. So, the required $R = 656.4/0.18 = 3650$.
$N = R/m = \mathbf{1825}$.
(b) We have $m = 2$ so $d \sin\theta = 2\lambda$. Noting that $d = 2.00\ \mu$m, we have
$\sin\theta = 2\lambda/d = \{0.65645,\ 0.65627\}$ and
$\theta = \{\mathbf{41.030^\circ},\ \mathbf{41.016^\circ}\}$ and $\Delta\theta = \mathbf{0.014^\circ}$.
Google "Harold Urey."

P51. **Thickness of Human Hair.** Although we have discussed single-slit diffraction only for a slit, a similar result holds when light bends around a straight, thin object, such as a strand of hair. In that case, a is the width of the strand. From actual laboratory measurements on a human hair, it was found that when a beam of light of wavelength 632.8 nm was shone on a single strand of hair, and the diffracted light was viewed on a screen 1.25 m away, the first dark fringes on either side of the central bright spot were 5.22 cm apart. How thick was this strand of hair?

The diffraction pattern for a hair is the same as for a slit of equal width. The separation of the dark fringes is twice the distance from center to either fringe, so
$\sin\theta = (2.61)/\sqrt{2.61^2 + 125^2} = 0.02088$. Since $m = 1$,
$a \sin\theta = \lambda \quad \Rightarrow \quad a = (0.6238\ \mu\text{m})/0.02088 = \mathbf{29.88\,\mu m}$. Since the first dark fringes on either side are separated

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P58 Opt. The intensity of light in the Fraunhofer diffraction pattern of a single slit is

$$I = I_0 \left(\frac{\sin\gamma}{\gamma}\right)^2$$

where

$$\gamma = \frac{\pi a \sin\theta}{\lambda}$$

(a) Show that the equation for the values of at which is a maximum is $\tan\gamma = \gamma$. (b) Determine the three smallest positive values of that are solutions of this equation. (Hint:You can use a trial-and-error procedure. Guess a value of and adjust your guess to bring $\tan\gamma$ closer to γ. A graphical solution of the equation is very helpful in locating the solutions approximately, to get good initial guesses.)

(a) Standard calculus trick - take $dI/d\gamma$ and set to zero.
$dI/d\gamma = I_0(2)(\sin\gamma/\gamma)(\gamma\cos\gamma - \sin\gamma)/\gamma^2$.
Only two factors can be zero:
If $\sin\gamma = 0$, then $I = 0$, so this which clearly corresponds to minima.
If $(\gamma\cos\gamma - \sin\gamma) = 0$, then $\sin\gamma$ is clearly non-zero, so these can't correspond to minima; hence they correspond to maxima.
$\gamma\cos\gamma = \sin\gamma \quad \Rightarrow \quad \gamma = \tan\gamma$.
(b) From a rough sketch, we see that solutions are at $\gamma = 0$, and γ values a bit less than $\{3\pi/2, 5\pi/2, 7\pi/2,\ ...\} = \{4.6, 7.8, 10.9,\ ...\}$. By inspection, I for $\gamma = 0$ is a maximum, provided we define this as the limit as $\gamma \to 0$. Refining the other values, we see they are at
$\gamma = \{4.4934, 7.7253, 10.9041,\ ...\}$.
This is an example of a *transcendental equation*, which can only be solved numerically (although there are more efficient methods).
