Mat 1348: Set Theory II: Basic Definitions

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For further examples and illustration of the concepts below, see lectures in class, the appropriate sections of Rosen's text, and the Supplementary Exercises (with solutions) on Virtual Campus. There is a section on functions (Section 5), on relations (Section 6), and equivalence relations (Section 7). As described in class (and Rosen, p.41 on "quantifiers") we write $\forall x$ for "for all x" and $\exists x$ for "there exists an x".

1 Injective, Surjective, Bijective Functions

Definition 1.1

- A function $f : A \to B$ is *injective* if it satisfies:

  $\forall x_{\in A} \forall x'_{\in A} [f(x) = f(x') \to x = x']$ equivalently $\forall x_{\in A} \forall x'_{\in A} [x \neq x' \to f(x) \neq f(x')]$. This says: distinct inputs from A map (via f) to distinct outputs.

- A function $f : A \to B$ is *surjective* if it satisfies: $Range(f) = B$, i.e.

  $\forall y_{\in B} \exists x_{\in A} [\ f(x) = y\ ]$. This says: for every codomain element $y \in B$, there is some $x \in A$ such that $y = f(x)$.

- A function $f : A \to B$ is *bijective* if it is injective and surjective. Bijections are also referred to as *one-one correspondences* between elements of A and B.

Important Notation/Terminology: Rosen's and some other American textbooks use the terminology *one-one* for *injective* and *onto* for *surjective*. The terminology injective/surjective/bijective is the standard terminology in advanced mathematics and agrees with the French terminology. We will *not* use Rosen's terminology in our section.

We gave many examples in class. The key facts are:

1. To show a function $f : A \to B$ is *not* injective, you must give an explicit counterexample: you must show: $\exists x \in A\, \exists x' \in A$ such that $f(x) = f(x')$ but $x \neq x'$.

2. To show $f : A \to B$ is surjective, you must show directly that the range of f is B, i.e. you must show how if you start with an element $y \in B$, you can find an $x \in A$ such that $f(x) = y$.

3. To show $f : A \to B$ is *not* surjective, you must find an element of the codomain which is not "hit" by f; i.e. find an element $y \in B$ so that $y \neq f(x)$ for any $x \in A$.

4. As shown in class, bijective functions $f : A \to B$ have inverse functions $f^{-1} : B \to A$ such that $f(f^{-1}(y)) = y$, for all $y \in B$ and such that $f^{-1}(f(x)) = x$ for all $x \in A$. f^{-1} is a well-defined "reverse process", so that if f maps $x \mapsto y$, then f^{-1} maps $y \mapsto x$. This sets up a one-to-one correspondence between elements of A and of B.

   NOTE: $f^{-1}(x)$ has nothing to do with $1/f(x)$, even for real-valued functions and those x for which $f(x) \neq 0$.

Examples 1.2

1. The quadratic $f(x) = x^2 + 1$, where we consider $f : \mathbb{R} \to \mathbb{R}$, is not injective, since $f(1) = f(-1) = 2$, but $1 \neq -1$.

2. The above function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$ is *not* surjective (draw the graph). For example, notice $f(x) \geq 1$. So in particular, $0 \notin Range(f)$, nor is any negative number in the range of f. BUT, if we redefine the codomain of f to say: consider $f : \mathbb{R} \to \mathbb{R}^{\geq 1}$, where $\mathbb{R}^{\geq 1} = \{x \in \mathbb{R} \mid x \geq 1\}$ and $f(x) = x^2 + 1$, then f is surjective: given $y \in \mathbb{R}, y \geq 1$, solve $y = x^2 + 1$ for x to get $x = \sqrt{y-1}$. Notice that this square root is well-defined since $y \geq 1$. Now notice that given the y, we can calculate $f(x) = (\sqrt{y-1})^2 + 1 = y$, so f is surjective. Notice that $f : \mathbb{R} \to \mathbb{R}^{\geq 1}$ is still not injective. Why?

3. To find the inverse function of a bijective function $y = f(x)$, in many cases of simple functions (for example in calculus), you can solve for x in terms of y, as we did above. Of course, f^{-1} is only well-defined when we know f is bijective. If you solve the equation $y = f(x)$ for x, this will give $x = f^{-1}(y)$. To find $f^{-1}(x)$, substitute x for y in the expression for $f^{-1}(y)$. (**Note**: This method of finding the inverse f^{-1} of a bijective function f works well for simple expressions where you can "solve" for x in terms of y, when you know $y = f(x)$. In general, there may not be a nice simple formula for f^{-1}, given a bijective f.)

   For example, restrict the domain and codomain of f in (2) to get: $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 1}$, where $f(x) = x^2 + 1$. Now this f *is* bijective (draw the graph). For the inverse, given $y \in \mathbb{R}^{\geq 1}$, as above solve to get $x = f^{-1}(y) = \sqrt{y-1}$. Hence, as a function of the variable x, we get: $f^{-1}(x) = \sqrt{x-1}$. So the inverse function $f^{-1} : \mathbb{R}^{\geq 1} \to \mathbb{R}^{\geq 0}$ "reverses" the action of f, and is well-defined (since the domain consists of all $x \geq 1$. The inverse equations are easily checked.

Definition 1.3 (Inverse Functions) Given a function $f : A \to B$, we say a function $h : B \to A$ is an *inverse* function of f if $h \circ f = id_A$ and $f \circ h = id_B$. Equivalently, h is an inverse function of f if $h(f(x)) = x$, for all $x \in A$ and $f(h(y)) = y$, for all $y \in B$.

For example, if $f : A \to B$ is bijective, then f^{-1} exists and is an inverse of f.

Proposition 1.4 . *Let $f : A \to B$ be a function.*

(i) *An inverse function of f, if it exists, is always unique.*

(ii) *A function f is bijective iff it has an inverse function. (note by part (i), the inverse function is unique).*

Proof. (i) Suppose there are two inverses $h_1, h_2 : B \to A$ of $f : A \to B$. Then $h_1 \circ f = id_A$ and $f \circ h_1 = id_B$ and $h_2 \circ f = id_A$ and $f \circ h_2 = id_B$. Then we have:

$$h_1 = h_1 \circ id_B = h_1 \circ (f \circ h_2) = (h_1 \circ f) \circ h_2 = id_A \circ h_2 = h_2.$$

(ii) ($\Rightarrow$) direction shown in class. Conversely, for the ($\Leftarrow$) direction, suppose f has an inverse function h . Then we must prove f is bijective. **Injective:** suppose $f(x) = f(x')$. Then applying the inverse, $h(f(x)) = h(f(x'))$, so $id_A(x) = id_A(x')$, i.e. $x = x'$. **Surjective.** Given $y \in B$, let $x = h(y) \in A$, which exists since the inverse function h exists. Then $f(x) = f(h(y)) = (f \circ h)(y) = id_B(y) = y$. $\square$

2 Relations

Definition 2.1 A *(binary) relation* from A to B is a subset $R \subseteq A \times B$. An *n-ary relation* is a subset $R \subseteq A_1 \times \cdots \times A_n$. A *(binary) relation on* A is a subset $R \subseteq A \times A$.

Notation: $(a, b) \in R$ is also denoted aRb or sometimes $R(a, b)$.

We will give many examples of relations in class (and also see the book and the Supplemental Exercises). Let us give some basic definitions, from Rosen's book. We mostly focus on relations on A, rather than the slightly more general case of relations from A to B.

Definition 2.2 Let R to be a relation on A, i.e. $R \subseteq A \times A$.

1. R is *reflexive* if for all $a \in A$, aRa, i.e. $(a, a) \in R$, for all $a \in A$.
2. R is *irreflexive* of for all $a \in A$, it's not true that aRa; that is, R is irreflexive if for all $a \in A, (a, a) \notin R$. Still another way to say this is that R is irreflexive iff the complement relation $(A \times A) - R$ is reflexive.
3. R is *symmetric* if for all $a, b \in A$, aRb implies bRa.
4. R is *transitive* if for all $a, b, c \in A$, aRb and bRc implies aRc.
5. R is *antisymmetric* if for all $a, b \in A$, aRb and bRa implies $a = b$.

Examples 2.3 The easiest examples are (i) relations on finite sets A and (ii) natural examples from mathematics ; but in the textbook and in class we will see other kinds of relations.

1. For example, using finite sets, say $A = \{1, 2, 3, 4\}$, it's easy to find examples of relations satisfying some of the above properties but not others. For example:
   (a) $R_1 = \{(1, 1), (1, 2)\}$ is not reflexive, not irreflexive, not symmetric, but it is transitive, and it is antisymmetric, since "$(1, 2) \in R_1$ and $(2, 1) \in R_1$" is false, thus anything follows!
   (b) $R_2 = \{(1, 1), (1, 2), (2, 1)\}$ is not reflexive, not irreflexive, it *is* symmetric but not transitive (since $(2, 1)$ and $(1, 2) \in R_2$, but $(2, 2) \notin R_2$.
   (c) $R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ is symmetric and transitive. It is not reflexive, nor irreflexive.
   (d) $R_4 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ is symmetric and transitive but still not reflexive (since it's missing $(4, 4)$).

(e) $R_5 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1)\}$ is reflexive, symmetric, and transitive (see equivalence relations, below).

2. Examples from mathematics.

   (a) Take $A := \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Then the relation $a < b$ is irreflexive, transitive, and not symmetric.

   (b) Take $A := \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Then the relation $a \leq b$ is reflexive, transitive, and antisymmetric (the latter because $a \leq b$ and $b \leq a$ implies $a = b$.)

   (c) Take A to be the set of lines in the coordinate plane $\mathbb{R}^2$. Consider the relation $\ell R \ell'$ iff $\ell || \ell'$ (i.e. the lines are in relation R iff they are parallel to each other). This is reflexive, symmetric, transitive.

   (d) Consider the "equality relation" on any set A: aRb iff $a = b$. This is reflexive, symmetric, transitive.

   (e) For 2×2 matrices M, N over $\mathbb{R}$, define MRN iff $det(M) = det(N)$. This is reflexive, symmetric, transitive.

   (f) On $\mathbb{Z}^+$, define $m|n$ to be the relation that "m evenly divides n" . Then clearly *divides* is reflexive, transitive. It is not symmetric: for example $2|6$ but $6 \not| 2$ (i.e. 6 does not divide 2). Exercise: prove that over $\mathbb{Z}^+$, if $a|b$ and $b|a$, then $a = b$, so "divides" is antisymmetric (only over $\mathbb{Z}^+$, not over $\mathbb{Z}$.

3 Equivalence Relations and Partitions

Definition 3.1 An *equivalence relation* is a relation $R \subseteq A \times A$ that satisfies the following conditions: reflexive, symmetric, transitive:

Reflexive $\forall x_{\in A}(xRx)$

Symmetric $\forall x_{\in A} \forall y_{\in A}(xRy \Rightarrow yRx)$

Transitive $\forall x_{\in A} \forall y_{\in A} \forall z_{\in A}(\ (xRy \wedge yRz) \Rightarrow xRz\ )$

Definition 3.2 A *partition* of the set A is a family of subsets $\mathcal{P} = \{P_i \subseteq A \mid i \in I\}$ such that (i) $P_i \cap P_j = \emptyset$ if $i \neq j$ (disjointedness) and (ii) $\cup_{i \in I} P_i = A$ (we say $\mathcal{P}$ covers A).

We drew many examples of partitions in class: just take the set A and partition it into a family $\mathcal{P} = \{P_i\}$ of disjoint regions (think of a geographical map divided into physically disjoint regions or countries).

Definition 3.3 Let R be an equivalence relation on set A. The *equivalence class of x modulo R* is the set

$$[x]_R = \{y \in A \mid xRy\}$$

The main theorems about equivalence relations are the following:

Theorem 3.4 *Let R be an equivalence relation on the set A. Let $a, b \in A$. The following are equivalent:*

(i) *aRb, i.e. $\langle a, b \rangle \in R$.*

(ii) $[a]_R \cap [b]_R \neq \emptyset$.

(iii) $[a]_R = [b]_R$.

So let R be an equivalence relation on A. Then the above theorem says that if two R-equivalence classes overlap at all, then they are equal. Hence equivalence classes are either disjoint or they overlap, and in the latter case, they are equal. In fact, we have:

Corollary 3.5 *The R-equivalence classes form a partition of the set A:*

$$A = \cup\{\ [a]_R \mid a \in A\ \} = \cup\{P_a \mid a \in A\},\ \textit{where } P_a = [a]_R$$

Proof. The idea is that: it is easy to show that the P_a form a disjoint collection of sets whose union is a subset of A, by the theorem above. It actually equals A because every element of $a \in A$ is in exactly one of these partitions: namely $a \in [a]_R$ because this just says aRa, which is just reflexivity. □

Finally, this allows us to set up a bijection:

$$\boxed{\textit{Equivalence Relations } R \textit{ on set } A} \quad \cong \quad \boxed{\textit{Partitions of set } A}$$

The proof is as follows:

From left to right, we associate to an equivalence relation the partition given by its equivalence classes, i.e. $R \mapsto \mathcal{P} = \{[a]_R \mid a \in A\}$.

Conversely, given a partition $\mathcal{P}$, we associate the equivalence relation $R_{\mathcal{P}}$ on A as follows: $aR_{\mathcal{P}}b$ iff $a, b \in P_i$ for the same partition component P_i of $\mathcal{P}$. In words *a and b are $R_{\mathcal{P}}$-related iff they are in the same component of the partition $\mathcal{P}$.* As shown in class, this gives an equivalence relation and I leave as an exercise the fact that this correspondence sets up a bijection between equivalence relations R on A and partitions $\mathcal{P}$ of A.

Example 3.6 Partition the set $A = \{0, 1, 2, 3, 4\}$ as follows: $\mathcal{P} = \{\{0, 3\}, \{2, 4, 1\}\}$. Then the associated equivalence relation

$$R = \{\underbrace{(3,0),(0,3),(3,3),(0,0)}_{\text{Corresponds to } \{0,3\}}\ ,\ \underbrace{(2,4),(4,1),(2,1),(4,2),(1,4),(1,2),(1,1),(2,2),(4,4)}_{\text{Corresponds to } \{2,4,1\}}\ \}$$

Example 3.7 Same as above, except $A = \{0, 1, 2, 3, 4, 5\}$ and partition $\mathcal{P}' = \{\{0, 3\}, \{2, 4, 1\}, \{5\}\}$. Then the associated equivalence relation is $R' = R \cup \{(5, 5)\}$, i.e. just add the pair $(5, 5)$ to the R above. Why does this correspond to the partition $\mathcal{P}'$?