MAT 1341C W11 Final Solutions 12'5

1. Let $W = \{(x, y, z, w) \in \mathbf{R}^4 \mid xy + zw \geq 0\}$. Then,

A. $(0, 0, 0, 0) \in W$ but W is not closed under multiplication by scalars. × W is closed under mult'n by scalar

B. $(0, 0, 0, 0) \notin W$ but W is closed under addition. × $e_1, -e_2 \in W$ but $e_1 - e_2 \notin W$.

C. W is closed under addition but W is not closed under multiplication by scalars.

D. W is closed under addition and W is closed under multiplication by scalars.

E. W is not closed under addition but W is closed under multiplication by scalars.

F. None of the other statements is true.

2. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbf{R}$, $au + bv + cw = 0$ implies $a = b = c = 0$. ✓

II. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbf{R}$, $au + bv + cw = 0$ if $a = b = c = 0$. ×

III. A set $\{u, v, w\}$ of vectors is linearly independent if u is not a multiple of v, and u is not a multiple of w. ×

l.d., only 3 vecs

IV. $\{(1, 0, 1), (0, 1, 0), (1, 1, 1)\}$ spans $\mathbf{R}^3$. ×

V. $\{(1, -1), (1, 1)\}$ is linearly independent in $\mathbf{R}^2$. ✓

A. I & II

B. II & IV

C. I & IV

D. III & V

E. III & II

F. I & V

3. If three $n \times n$ matrices A, B and C satisfy $AB - BA = C$, then ABA is **always** equal to :

A. $A^2B - C$

B. $A^2B - CA$

C. $BA^2 + CA$

D. A^2B

E. $A^2B + AC$

F. $A^2B + BC$

$\Rightarrow BA = AB - C$

$\Rightarrow ABA = A^2B - AC$

or

$AB = BA + C$

$\Rightarrow ABA = BA^2 + CA$

4. Complete the following phrase to make a true statement:

"A homogeneous linear system of 2011 linear equations in 1231 unknowns..."

A. ... is always consistent.

B. ... always has a unique solution. ×

C. ... may be inconsistent. ×

D. ... which is consistent always has a unique solution. ×

E. ... which is consistent never has a unique solution. it could, if rank $A = 1231$, which is possible.

F. ... is never consistent.

$$\begin{array}{cc} & 1231 \\ 2011 & \left[\begin{array}{c|c} A & 0 \end{array}\right] \end{array}$$

5. Compute $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2011} = A$

$A^2 = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a+ba & b^2 \end{bmatrix}$

But $ba = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$

and $b^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = b$

$\therefore A^2 = A$

Hence $A^n = A, \forall n \geq 1$.

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2011 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2011 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 0 & 0 & 2011 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ F. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2011 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. The dimension of $S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}(\mathbf{R}) \mid a+d=0\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. S is not a vector space, so we cannot speak of its dimension.

$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle| a,b,c \in \mathbb{R} \right\} = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{M_3} \right\}$

and $aM_1 + bM_2 + cM_3 = 0 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$
$\Rightarrow a=b=c=0$.

Hence $\{M_1, M_2, M_3\}$ is a basis and so $\dim S = 3$

7. The matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ is not diagonalizable over the reals. Why?

A. Because A does not have any real eigenvectors   not true: $Ae_1 = 4e_1$

B. Because A does not have any real eigenvalues..   " $\lambda = 4$ is an evl

C. Because A does not have three distinct eigenvalues.   this is not necessary

D. Because A does not have three independent eigenvectors. ←

E. Because A is lower triangular.   × Many lower Δular matrices are diag'ble. eg I_3

F. Because 'diagonalizable' is too difficult to say very quickly. ×

$E_3 = \ker(A - 3I) = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has dim 1

+

$E_4 = \ker(A - 4I) = \ker \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ has dim 1

$2 < 3$

8. The set of vectors $\{(1,1,1), (1,-1,0), (1,1,-2)\}$ is an orthogonal basis of $\mathbf{R}^3$. Find (c_1, c_2, c_3) such that $(1,0,0) = c_1(1,1,1) + c_2(1,-1,0) + c_3(1,1,-2)$.

A. $(-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$

B. $(\frac{1}{2}, -\frac{1}{3}, \frac{1}{6})$

C. $(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{6})$

D. $(\frac{1}{9}, \frac{1}{4}, \frac{1}{36})$

E. $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$

F. $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}})$

$c_1 = \frac{(1,0,0)\cdot(1,1,1)}{\|(1,1,1)\|^2} = \frac{1}{3}$ So (E) is correct

$c_2 = \frac{(1,0,0)\cdot(1,-1,0)}{\|(1,-1,0)\|^2} = \frac{1}{2}$ (A (E))

$c_3 = \frac{(1,0,0)\cdot(1,1,-2)}{\|(1,1,-2)\|^2} = \frac{1}{6}$

9. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & x \end{bmatrix}$. For which value(s) of x is A invertible?

A. $x \neq -1$
B. $x \neq 1$
C. $x \neq 0$
D. $x = -1$
E. $x = 1$
F. $x \neq \pm 1$

$|A| \overset{\text{row 2}}{=} 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & x \end{vmatrix} = x+1 \quad \therefore \quad x+1 \neq 0$

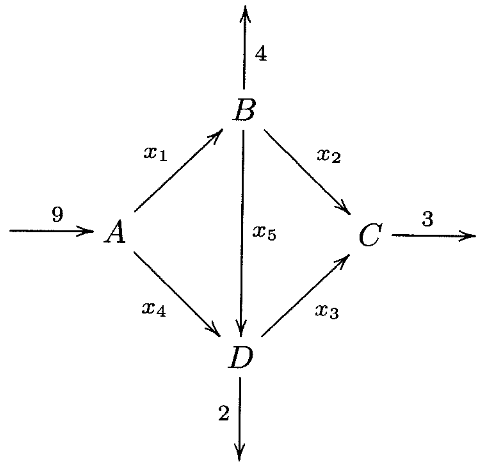
ie $x \neq -1$

10. Which of the following sets are linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{x, x^2\}$ ✓
$T = \{1, x, x^2, (1-x)^2\}$ ✗ $\quad (1-x)^2 = 1 - 2.(x) + 1\, x^2$
$U = \{1, 2\cos^2 x, 3\sin^2 x\}$ ✗ $\quad 1 = \frac{1}{2}(2\cos^2 x) + \frac{1}{3}(\sin^2 x)$

A. S and T.
B. S and U.
C. T and U.
D. S only.
E. T only.
F. S, T and U.

11. Consider the network of streets and intersections below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Intersection	Flow in	=	Flow out
A	9	=	$x_1 + x_4$
B	x_1	=	$4 + x_2 + x_5$
C	$x_2 + x_3$	=	3
D	$x_4 + x_5$	=	$2 + x_3$

$x_i \geq 0$ (1-way streets)

$x_i \in \mathbb{Z}$ (# of vehicles)

11 b). The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{ccccc|c} & & & s & t & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 9 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints at this point.)

$$\begin{aligned} x_1 &= 9 - s \\ x_2 &= 5 - s - t \\ x_3 &= -2 + s + t \\ x_4 &= s \\ x_5 &= t \end{aligned} \quad ; \; s, t \in \mathbb{R}$$

c) If $\overline{BD}$ were closed due to roadwork, find the maximum and minimum flow along $\overline{AB}$, **using your results from (b).**

$\overline{BD}$ closed $\iff x_5 = 0 \iff t = 0$.

Flow on $\overline{AB}$ is x_1 ; $x_1 = 9 - s$ (so $s \le 9$) ;

$x_2 \ge 0 \iff 5 - s \ge 0 \iff 5 \ge s$

$x_3 \ge 0 \iff -2 + s \ge 0 \iff s \ge 2$

$x_4 \ge 0 \iff s \ge 0$

$x_1 \ge 0 \iff 9 - s \ge 0 \iff 9 \ge s$

$\therefore 5 \ge s \ge 2$

$\therefore$ Max. value of x_1 is 7 and min value is 4

12. Let $U = \{(x, y, z, w) \in \mathbf{R}^4 \mid x + y + z - w = 0\}$.

a) Find a basis of U and give the dimension of U.

b) Find an orthogonal basis of U.

c) Find the best approximation to $(0, 1, 1, 1)$ by vectors in U.

a) $\left[\begin{array}{cccc|c} 1 & \overset{s}{1} & \overset{t}{1} & \overset{r}{-1} & 0 \end{array}\right]$; $(x_1, x_2, x_3, x_4) = (-s-t+r, s, t, r)$

$= s(-1,1,0,0) + t(-1,0,1,0) + r(1,0,0,1)$; $s, t, r \in \mathbb{R}$

$\therefore \{\underset{v_1}{(-1,1,0,0)}, \underset{v_2}{(-1,0,1,0)}, \underset{v_3}{(1,0,0,1)}\}$ is a basis of U.

Hence $\dim U = 3$.

b) $w_1 = v_1$

$w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 = (-1, 0, 1, 0) - \frac{1}{2}(-1, 1, 0, 0)$

$= (-\frac{1}{2}, -\frac{1}{2}, 1, 0)$.

Let $w_2' = (-1, -1, 2, 0)$

$w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2'}{\|w_2'\|^2} w_2'$

$= (1, 0, 0, 1) - \frac{(-1)}{2}(-1, 1, 0, 0) - \frac{(-1)}{6}(-1, -1, 2, 0)$

$= (1 - \frac{1}{2} - \frac{1}{6}, \frac{1}{2} - \frac{1}{6}, \frac{1}{3}, 1) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$

$\therefore \{\underset{u_1}{(-1,1,0,0)}, \underset{u_2}{(-1,-1,2,0)}, \underset{u_3}{(1,1,1,3)}\}$ is an orthog. basis of U.

c) $\mathrm{Proj}_U (0,1,1,1) = \frac{(0,1,1,1) \cdot u_1}{2} u_1 + \frac{(0,1,1,1) \cdot u_2}{6} u_2 + \frac{(0,1,1,1) \cdot u_3}{12} u_3$

$= \frac{1}{2} u_1 + \frac{1}{6} u_2 + \frac{5}{12} u_3 = \left(-\frac{1}{2} - \frac{1}{6} + \frac{5}{12}, \frac{1}{2} - \frac{1}{6} + \frac{5}{12}, \frac{2}{6} + \frac{5}{12}, \frac{15}{12}\right) = \left(-\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{15}{12}\right)$

12^{41}

13. $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$.

a) Compute $\det(A - \lambda I_3)$ and hence show that the eigenvalues of A are 7 and 1.

b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = x\}$.

c) Find a basis of $E_7 = \{x \in \mathbf{R}^3 \mid Ax = 7x\}$.

d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

e) Find an invertible matrix $Q \neq P$ such that $Q^{-1}AQ = \tilde{D}$ is also diagonal, and give this diagonal matrix $\tilde{D}$.

a) $|A - \lambda I_3| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{vmatrix} \overset{-R_1+R_3 \to R_3}{=} \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ -1+\lambda & 0 & 1-\lambda \end{vmatrix} = (\lambda-1) \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & -1 \end{vmatrix}$

$= (\lambda-1) \begin{vmatrix} 3-\lambda & 2 & 5-\lambda \\ 2 & 3-\lambda & 4 \\ 1 & 0 & 0 \end{vmatrix} = (\lambda-1) \begin{vmatrix} 2 & 5-\lambda \\ 3-\lambda & 4 \end{vmatrix} = (\lambda-1)\{8 - (\lambda^2 - 8\lambda + 15)\}$

$= -(\lambda-1)\{\lambda^2 - 8\lambda + 7\} = -(\lambda-1)(\lambda-1)(\lambda-7)$. Thus $|A - \lambda I_3| = 0 \iff \lambda = 1$ or $\lambda = 7$

b) $E_1 = \ker[A - I] = \ker \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \ker \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \{(-s-t, s, t) \mid s, t \in \mathbb{R}\}$

$\therefore \{(-1,1,0), (-1,0,1)\}$ is a basis for E_1

c) $E_7 = \ker[A - 7I] = \ker \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & -2 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \ker \begin{array}{c} \begin{array}{ccc} & & s \end{array} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \{(-3s, s, s) \mid s \in \mathbb{R}\}$ $\therefore$ $\{(-3,1,1)\}$ is a basis for E_7

d) Set $P = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$. Since $\dim E_1 + \dim E_7 = 3 = \dim \mathbb{R}^3$, P is invertible, and $P^{-1}AP = D$.

e) Set $Q = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $\tilde{D} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\det Q = \det P \neq 0$ so Q is invertible, and $Q^{-1}AQ = \tilde{D}$.

14. Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 2 & 0 & 6 \\ 1 & 2 & 4 & 2 \end{bmatrix}$$

a) Find a basis for the column space col(A) of A.

b) Give a complete geometric description of col(A).

c) Find a basis for the kernel, $\ker T$, of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x) = Ax, \quad x \in \mathbf{R}^4.$$

d) Compute $\dim(\ker T) + \dim(\operatorname{im} T)$.

a) C_1 C_3
$A \sim \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore$ $\{ \overset{v_1}{(1,1,1)}, \overset{v_2}{(2,0,4)} \}$

is a basis for col A.

b) Since $v_1 \times v_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 0 & 4 \end{vmatrix} = (4, -2, -2)$, col A is the plane through O with normal $(4,-2,-2)$. (or $(2,-1,1)$)

c) $\ker T = \ker A = \ker \begin{array}{c} \;\; s \;\;\;\;\;\; t \\ \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = \{(-2s-6t, 0, t, t) \mid s, t \in \mathbb{R}\}$
(A is the std matrix of T)

$\therefore$ $\{(-2,1,0,0), (-6,0,1,1)\}$ is a basis for $\ker T$.

d) im T = col A since A is the standard matrix of T. Hence

dim(im T) = dim col A = 2 (= rank A), &

dim im T + dim ker T = 2 + 2 = 4

(This is also given by the Conservation of dimension)

15. a) Let A be a real $n \times n$ matrix. Give 3 statements (in total) equivalent to

"$\det A = 0$",

one each in terms of:

(I) the rows of A

one of

The rows of A are l.d.

" do not span $\mathbb{R}^n$

" are not a basis of $\mathbb{R}^n$

(II) the rank of A

rank $A < n$.

(III) non-homogeneous systems $Ax = b$, for all $b \in \mathbf{R}^n$

There is $b \in \mathbb{R}^n$ st $Ax = b$ is not consistent.

or

If a homogeneous system $Ax = b$ is consistent, it will have ∞ly many solns.

15b) State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) $\begin{bmatrix} 6 & 0 \\ 1 & 5 \end{bmatrix}$ is diagonalizable.

True, because the 2×2 matrix has 2 distinct evals, namely 6 and 5.

$\boxed{\text{TRUE}}$

ii) If 0 is an eigenvalue of 4×4 matrix A, then A is invertible.

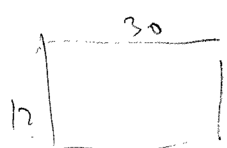
If 0 is an eval, $0 = \det(A - 0I) = \det(A)$.

(or if $v \neq 0$ is an evec with eval 0, $Av = 0$; but

A inv $\Rightarrow$ $A^{-1}(Av) = A^{-1}(0) \Rightarrow$ $v = 0$, a contradiction.)

$\boxed{\text{FALSE}}$

iii) The columns of a 12×30 matrix are always linearly dependent.



30 vectors in $\mathbb{R}^{12}$ are always dependent, since $30 > \dim \mathbb{R}^{12} = 12$.

$\boxed{\text{TRUE}}$

11pm

16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.

In what follows, A denotes an $n \times n$ matrix.

a) Prove that if v and w are eigenvectors of A corresponding to distinct eigenvalues, then $\{v, w\}$ is linearly independent.

Suppose $Av = \lambda v$, $Aw = \mu w$, and $\lambda \neq \mu$. Suppose now that $av + bw = 0$ (1) for scalars $a, b \in \mathbb{R}$.

Then $a Av + b Aw = 0 \Rightarrow a\lambda v + b\mu w = 0$. (2)

(2) - λ(1) gives $b\mu w - b\lambda w = 0$, or $b(\mu-\lambda) w = 0$.

But $(\mu-\lambda) \neq 0$ and $w \neq 0$ (it is an evec.), hence $b = 0$.

Then, (1) $\Rightarrow v = 0$ since $v \neq 0$ as well, being itself an evec of A.

b) Prove that if all the eigenvalues of A are non-zero, then A is invertible.

If A were not invertible, $\exists\, v \in \mathbb{R}^n$, $v \neq 0$ st $Av = 0$. But then $Av = 0v$, so $v (\neq 0)$ is an evec of A with eval 0. But all evals are not zero, so A must be invertible.

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