

1.2. Echelon forms : Elementary Row Operations

Def: A leading entry of a row is the left-most non-zero entry.

A row with only zeros has no leading entry.

Def: A matrix is in row echelon form (AKA: REF) if the following 3 conditions hold:

- 1) Any rows consisting of only zeros are at the bottom.
- 2) Each leading entry of a row is to the left of the leading entry of the row below it.
- 3) All entries in a column below a leading entry are zero.

Ex

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \text{REF} \quad , \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix} - \text{REF}$$

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Examples of Matrices not in RREF.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 3 \end{pmatrix} - \text{Satisfies } 2 \text{ : } 3.$$

- 1 not satisfied.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \text{Satisfies } 1$$

- ~~Doesn't~~ Doesn't satisfy 2 or 3.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \text{Satisfies } 1 \text{ : } 3$$

- Doesn't satisfy 2.

- However, if both 1 : 2 are satisfied, 3 is automatically satisfied.

- That is, if all rows of zeros are at the bottom and ^{each} ~~all~~ leading entry of a row is to the left of the leading entry of the row below it, then it must happen that all ~~leading~~ entries below a leading entry are zero.

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1.7.2 - Row reduced echelon form.

Def: If a matrix is in row echelon form (REF) and satisfies the additional 2 conditions, the matrix is said to be in row-reduced echelon form (RREF).

1) The leading entry of every row is 1.

2) The entries above a leading entry are all zero.

$$\text{Ex: } \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \text{RREF}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \text{RREF}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \text{REF not RREF}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \text{RREF}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} - \text{not in REF so can't be RREF.}$$

1.2.3 - Elementary Row Operations

The primary method of manipulating matrices is through elementary row operations

Elementary Row Operations

1) Interchange any two rows.

$$\text{Ex}_{//} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) Multiply a row by any non-zero number

$$\text{Ex}_{//} \quad \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 = \left(\frac{1}{2}\right) R_1 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 = (5) R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) Add a multiple of one row to another row

$$\text{Ex}_{//} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} R_2 = R_2 - 2R_1 \quad \begin{bmatrix} 1 & 2 & 3 \\ 4-2 & 5-4 & 6-6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

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We can use elementary matrix operations to transform a matrix into REF and RREF.

Theorem: Every matrix can be reduced to one and only one matrix in reduced row echelon form (RREF).

Ex. Use elementary row operations to transform the following matrix into RREF.

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 2 & 1 & 8 & 0 \\ 4 & 0 & 12 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 = R_2 + (-2)R_1 \\ R_3 = R_3 + (-4)R_1 \end{array} \quad \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & -1 & -2 & -2 \\ 4 & 0 & 12 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -4 & -8 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 = (-1)R_2 \\ R_3 = (-1)R_3 \end{array} \quad \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 8 & 3 \end{bmatrix} \quad \begin{array}{l} R_1 = R_1 - R_2 \\ R_3 = R_3 - 4R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 8 & 3 \end{bmatrix} \quad R_3 = R_3 + (-4)R_2 \quad \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -5 \end{bmatrix} \quad R_3 = \left(-\frac{1}{5}\right)R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 = R_1 + R_3 \\ R_2 = R_2 - 2R_3 \end{array} \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{RREF}$$

HW: #1.2 - #1-12, 13, 14 and 18 (for 13, 14).