1) Let $f(x) = \cos x$. Find $f^{(1291)}(x)$

Sol: $f(x) = \cos x$ | $f'''(x) = \sin x$

$f'(x) = -\sin x$ | $f^{(4)}(x) = \cos x$

$f''(x) = -\cos x$

So: $f^{(1291)}(x) = \sin x$.

($1291 = \underbrace{(4 \times 322)} + 3$

after $f^{(4\times 322)}(x) = \cos x = f^{(1288)} = \cos x$

so need extra 3 more derivative from $f^{(1288)}$ to $f^{(1291)}$)

2) Find all critical points for the function

$f(x) = (x-5)\, x^{2/3}$

$f'(x) = \frac{2}{3}(x-5)^{-\frac{1}{3}}(1) = \frac{2}{3\sqrt[3]{x-5}} = 0$

has no solution

$\Rightarrow$ no critical point. (None of these)

3) A side of a square is increasing at the rate of 2 feet per minute. Find the rate at which the area is increasing when the side is 7 feet.

Sol: We know:

$\frac{ds}{dt} = 2\,ft/min$, we want $\frac{dA}{dt}$

$A = s^2$. Take derivative both side in term of time

$\frac{dA}{dt} = 2s \cdot \frac{ds}{dt} \Rightarrow \frac{dA}{dt} = 2(7) \cdot 2 = 28$

$\Rightarrow$ the area is increasing at 28 square feet per min.

4) Given $f(x) = -\frac{8}{x}$. choose correct statement

$f'(x) = -\left[\frac{8' \cdot x - x' \cdot 8}{x^2}\right] = +\left[\frac{8}{x^2}\right]$ $\left(\frac{u}{v}\right)' = \frac{u'v \cdot}{v}$

$f''(x) = +\left[\frac{8' \cdot x^2 - (x^2)' \cdot 8}{x^4}\right] = +\left(-\frac{16x}{x^4}\right) = -\frac{16}{x^3}$.

So $f''(x) = \frac{-16}{x^3}$.

- Note: • If $x \in (-\infty; 0)$, then $f''(x) > 0 \Rightarrow$ concave up
  - • If $x \in (0, +\infty)$, then $f''(x) < 0 \Rightarrow$ concave down.

$\Rightarrow$ correct statement: f is concave up on $(-\infty$

Short answer:

1) Find the equation of tangent line to the curve

$y^3 - xy^2 + \cos(xy) = 2$. at $(0,1)$

First, equation of tangent line is: $y = k(x - x_0) + y$

where $k = \text{slope} = \frac{dy}{dx}$ = first derivative.

So we need to find $\frac{dy}{dx}$: Take derivate both sides.

Note: $(\cos x)' =$ ... $(\cos u)' = -u'$

$3y^2 \cdot \frac{dy}{dx} - \left(y^2 + 2xy \cdot \frac{dy}{dx}\right) + (xy)'(-\sin(xy)) = 0$

$\Leftrightarrow 3y^2 \cdot \frac{dy}{dx} - y^2 - 2xy \cdot \frac{dy}{dx} + y + x \cdot \frac{dy}{dx} \cdot \sin(xy) = 0$

$\Leftrightarrow 3y^2 \cdot \frac{dy}{dx} - 2xy \cdot \frac{dy}{dx} - x\sin(xy) \cdot \frac{dy}{dx} = y^2 - y$

$\Rightarrow \frac{dy}{dx}\left(3y^2 - 2xy - x\sin xy\right) = y^2 - y$

$\Rightarrow \frac{dy}{dx} = \frac{y^2 - y}{3y^2 - 2xy - x\sin xy}$.

$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1^2 - 1}{3(1)^2 - 2(0)(1) - 0\sin(0.1)} = 0$

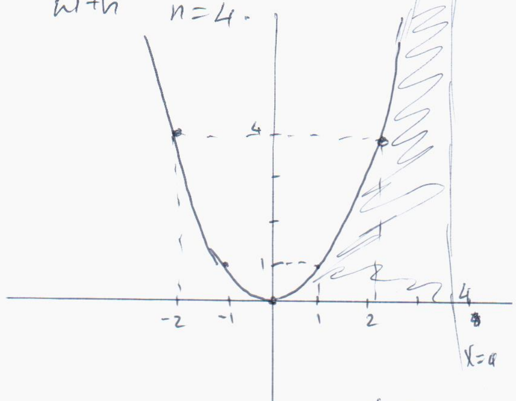
$\Rightarrow$ Equation of tangent line is:

$Y = y_0 = 1$.

$\Rightarrow y = 1$ is the equation of tangent line

Note: In case $\frac{dy}{dx} = k \neq 0$, then plug the value $k = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} \to$ given, and write equation of tangent line with k, (x_0, y_0).

2> Trapezoidal rule $f(x)=x^2$, x-axis and the line $x=4$ with $n=4$.



$f(x)=x^2$. we need to calculate the area only from 0 to 4.

$$\int_0^4 f(x) = \cancel{b-a} \ \tfrac{h}{2}\left[f(x_0)+2f(x_1)+\dots+f(x_n)\right]\frac{h}{2}$$

where $h = \frac{b-a}{n}$:

(Please check the formula, I don't remember)

$h = \frac{4-0}{4} = 1$

$n=4$:

$$\int_0^4 x^2 = \frac{h}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right]$$

single single.

we have: $h=1$; $x_0=0$; $x_1=x_0+h=1$
$x_2=x_1+h=2$
$x_3=x_2+1=3$
$x_4=x_3+1=4$.

$f(x)=x^2$

$f(x_0)=f(0)=0^2=0$
$f(x_1)=f(1)=1^2=1$
$f(x_2)=f(2)=2^2=4$
$f(x_3)=f(3)=3^2=9$
$f(x_4)=f(4)=4^2=16$

$$\Rightarrow \int_0^4 x^2 = \frac{1}{2}\left[0+2(1)+2(4)+2(9)+16\right]=\frac{1}{2}(44)=22.$$

$$\text{Area} = \int_{-2}^{0}|(x^3-4x)|dx+\int_0^2|(x^3-4x)|dx$$

$$=\left|\frac{x^4}{4}-2x^2\Big|_{-2}^{0}\right| + \left|\frac{x^4}{4}-2x^2\Big|_0^2\right| = \dots$$

absolute

This part is extra problem I found

① Find all the critical point for the funct

$f(x) = 2(x-3)\cdot x^{\frac{1}{2}}$

$\cancel{f'(x) = 2\cdot\frac{1}{2}(x-3)^{-\frac{1}{2}}(1)(2)^{\frac{1}{2}} + (\frac{1}{2}x^{-\frac{1}{2}})(2)(x-3)}$

$\cancel{= \frac{1\cdot\sqrt{x}}{\sqrt{x-3}} + \frac{2(x-3)}{2\sqrt{x}}}$

$\cancel{f'(x) = \frac{x+(x-3)\cdot 2\sqrt{x-3}}{\sqrt{x}(\sqrt{x-3})}} = \frac{x+(x-3)\sqrt{x-3}}{\sqrt{x}\cdot\sqrt{x-3}}.$

$\cancel{f'(x)=0}$

$$f'(x) = 2\left[(x-3)'\cdot x^{\frac{1}{2}} + (x^{\frac{1}{2}})'(x-3)\right]$$
$$= 2\left(x^{\frac{1}{2}} + \frac{1}{2}(x-3)\cdot x^{-\frac{1}{2}}\right]$$
$$f'(x) = 2\left(\sqrt{x} + \frac{x-3}{2\sqrt{x}}\right) = 2\left(\frac{2x+x-3}{2\sqrt{x}}\right)$$
$$f'(x) = \frac{2(3x-3)}{2\sqrt{x}} = \frac{3x-3}{\sqrt{x}}$$
$$f'(x)=0 \Leftrightarrow 3x-3=0 \Rightarrow x=1$$

• So $x=1$ ~~as f(1)~~ is critical point.

② Use trapezoidal rule with $n=4$ to approximate:

$n=4$

$$\int_0^{\pi} \sin x \, dx.$$

$$\int_0^{\pi}(\sin x)\,dx: \ \frac{b-a}{n} = \frac{\pi}{4} = h$$

$x_0=0$; $x_1 = x_0+h = \frac{\pi}{4} \Rightarrow \cancel{f(x_0)=f(0)=0}$

$x_2 = \frac{\pi}{4}+h = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$ ~~as f(x)~~

$x_3 = \frac{\pi}{2}+h = \frac{\pi}{2}+\frac{\pi}{4} = \frac{3}{4}\pi$

$x_4 = \pi$.

$f(x_0) = \sin 0 = 0$
$f(x_1) = f(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
$f(x_2) = f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$
$f(x_3) = \sin(\frac{3}{4}\pi)$
$f(x_4) = \sin(\pi) = 0$

$$\Rightarrow \int_0^{\pi}\sin x\,dx = \frac{\pi}{4}\left(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right) = .$$

3> Find the area of the region bounded by the gr
$f(x) = x^3-6x$ and $g(x) = -2x$

• First, find intercept point: $x^3-6x = -2x$
$\Leftrightarrow x^3-4x = 0 \Leftrightarrow x(x^2-4) = 0$
$\Leftrightarrow x(x-2)(x+2) = 0$
$\Rightarrow x=0;\ x=-2;\ x=2.$

⟸ continue

Extra work:

② Find the volume of the solid formed by revolving the region bounded by $y = x^2$; $y^2 = 8x$, about x-axis.

* Find the intersect point:

$$\begin{cases} y = x^2 \\ y^2 = 8x \end{cases} \Leftrightarrow \begin{cases} y^2 = x^4 \\ y^2 = 8x \end{cases}$$

$\Rightarrow \quad x^4 = 8x \Leftrightarrow x^4 - 8x = 0$

$\Leftrightarrow x(x^3 - 8) = 0 \Rightarrow x = 0;\ x = 2$

$\cancel{\Leftrightarrow x[(x)^3 - 2^3] = 0}$

$\cancel{= x(x-2)(x^2+2x+4) = 0}$

- Please look the book of this section, so you can learn how to do this problem.

Sol:

$$V = \pi \int_0^2 (8x - x^4)\,dx = \pi\left[8 \cdot \frac{x^2}{2} - \frac{1}{5}x^5\right]\Bigg|_0^5$$

$$V = \frac{48}{5}\pi.$$

This problem from text book page 332, example 3.

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③ Make sure you look this problem from text book page 162/example 2.

- Water is pouring into a conical tank at the rate 4 cubic feet per minute. If the height of the tank is 12 feet and the radius of its circular opening is 6 feet, how fast is the water level rising when the water is 4 feet deep.

Sol: Given $\frac{dV}{dt} = 4\ ft^3/min$

Find $\frac{dh}{dt}\Big|_{h=4}$.

The height of the tank is 12 feet and the radius is $r = 6$: $\frac{r}{6} = \frac{h}{12} \Rightarrow r = \frac{h}{2}$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi h^3}{12}$$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt} \Leftrightarrow 4 = \frac{\pi (4)^2}{4} \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi}.$$

Extra problems:

Find the equation of tangent line to the graph of $x^2 + 3y^2 = 4$ at $(1;1)$.

Sol: Equation of tangent line is: $y - y_0 = k(x - x_0)$ where $m = \frac{dy}{dx}$ at $(1;1)$.

$\Rightarrow 2x + 6y \cdot \frac{dy}{dx} = 0$ (take derivative both sides)

$\Leftrightarrow \frac{dy}{dx} = -\frac{x}{3y}$. At $(1;1)$, $\frac{dy}{dx} = -\frac{1}{3}$.

$\Rightarrow$ equation: $y - 1 = -\frac{1}{3}(x-1) \Leftrightarrow y = -\frac{1}{3}x + \frac{4}{3}$.

② Look this problem from text book (or google it)

The formular for the volume of a tank is $V = 2\pi r^3$, where r is the radius of the tank. If the radius is increasing at the rate of $\frac{3}{2}$ feet/min, find the rate which volume is ↗ when radius is 3 feet.

③ Given $f(x) = 10 - \frac{16}{x}$. Find all value of **c** in interval $(2, 8)$ such that $f'(c) = \frac{f(8) - f(2)}{8 - 2}$.

Sol: $f(8) = 10 - \frac{16}{8} = 8$, $f(2) = 2$ $\Big\rangle\ f'(c) = \frac{8-2}{8-2} = 1$.

$f(x) = 10 - \frac{16}{x} \Rightarrow f(c) = 10 - \frac{16}{c}$

$\Rightarrow f'(c) = 0 + \frac{16}{c^2} = 1 \Leftrightarrow 16 = c^2 \Rightarrow c = \pm 4$

④ Find y'' at $(2,1)$ if $2x^2y - 4y^2 = 4$

mean need to find $\frac{d^2y}{dx^2}$:

$4xy + 2x^2\frac{dy}{dx} - 8y \cdot \frac{dy}{dx} = 0$

$\Leftrightarrow 4xy = (8y - 2x^2)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{4xy}{8y - 2x^2}$.

Do 1 more time derivative ....

Final test question:

① Evaluate $\int_1^4 3x\sqrt{4-x^2}\,dx$

Sol: Let $u=\sqrt{4-x^2}$, $u^2=4-x^2$

$\Rightarrow \quad 2u\,du = -2x\,dx$

$-u\,du = x\,dx$

$\Rightarrow \int 3x\cdot\sqrt{4-x^2} = \int 3\cdot\sqrt{4-x^2}\cdot x\,dx$

$= 3\int u\cdot(-u\,du) = -3\int u^2du = -3\cdot\frac{u^3}{3}$

$= -u^3 = -\left(\sqrt{4-x^2}\right)^3$.

$\Rightarrow \int_1^4 3x\sqrt{4-x^2}\,dx = -\left[\left(\sqrt{4-x^2}\right)^3\right|_1^4$

$= -\left[\underbrace{\sqrt{(4-4^2)^3}}_{\text{undefined}} - \sqrt{(4-1^2)^3}\right)$

since $\sqrt{4-16}=\sqrt{-12}$: No

$\Rightarrow$ answer: None of these.

② Evaluate the integral:

$A=\int_0^1\left(\frac{1}{5}+\sin^2(\pi x)\cos^3(\pi x)\right)dx.$ ***********

$A=\int_0^1\frac{1}{5}dx+\int_0^1\sin^2(\pi x).\cos^2(\pi x).\cos(\pi x)\,dx$

Let $u=\sin\pi x$

$du=\pi\cos\pi x\,dx$ so $\cos(\pi x)dx=\frac{du}{\pi}$.

Since $u=\sin(\pi x)$, $u^2=\sin^2\pi x$

$\Rightarrow 1-u^2=1-\sin^2(\pi x)=\boxed{\cos^2(\pi x)}$

$A=\frac{1}{5}x\Big|_0^1+\int u^2(1-u^2)\cdot\frac{du}{\pi}$

$A=\frac{1}{5}(1-0)+\frac{1}{\pi}\int(u^2-u^4)\,du$

$A=\frac{1}{5}+\frac{1}{\pi}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)$

$A=\frac{1}{5}+\frac{1}{\pi}\left[\frac{1}{3}(\sin\pi x)^3\Big|_0^1-\frac{1}{5}(\sin\pi x)^5\Big|_0^1\right]$

$A=\frac{1}{5}+\frac{1}{\pi}\left[\frac{1}{3}\left[(\sin\pi)^3-(\sin 0)^3\right]-\frac{1}{5}\left[(\sin\pi)^5-(\sin 0)^5\right]\right]$

$A=\frac{1}{5}+\frac{1}{\pi}\left[\frac{1}{3}\cdot 0-0\right]-\frac{1}{5}(0-0)$

$\Rightarrow \boxed{A=\cancel{15}\ \frac{1}{5}}$ is an answer.

③ Evaluate integral $\int\frac{x+1}{x^2-5x+6}dx$.

$\frac{x+1}{x^2-5x+6}=\frac{x+1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}$.

$\frac{x+1}{(x-2)(x-3)}=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$\Rightarrow x+1=(A+B)x-(3A+2B)$

$\Rightarrow \begin{cases}A+B=1\\ -(3A+2B)=-1\end{cases} \Leftrightarrow \begin{cases}A=-3\\ B=\quad B=4\end{cases}$

$\Rightarrow \frac{x+1}{(x-2)(x-3)}=\frac{-3}{x-2}+\frac{4}{x-3}$

$\Rightarrow \int\frac{x+1}{(x-2)(x-3)}dx=\int\frac{-3}{x-2}dx+\int\frac{4}{x-3}dx$

$=-3.\ln|x-2|+4\ln|x-3|$

$=\ln|x-2|^{-3}+\ln|x-3|^4$

$=\ln\left[\left(|x-2|^{-3}\right).\left(x-3\right)^4\right]$

$=\ln\left|\frac{(x-3)^4}{(x-2)^3}\right|+C.$

④ Evaluate $\lim_{x\to0}\frac{3e^{\frac{x}{3}}-(3+x)}{x^2}=\frac{0}{0}$

$\xrightarrow{L-H}\lim_{x\to0}\frac{e^{\frac{x}{3}}-1}{2x}=\frac{0}{0}\xrightarrow{L-H}\lim_{x\to0}\frac{\frac{1}{3}e^{\frac{x}{3}}}{2}$

⑤ Using the trapezoidal rule with $n=4$ to approximate:

$$\int_3^7 (x^2-10x+21)\,dx.$$

$$h = \frac{7-3}{4} = 1.$$

I think you know how to do this question now.

ans: -10.67.

⑥ Set up the integral needed to find the volume of the solid formed when the graph of region bounded by $y=\sqrt{25-x^2}$ and $y=3$ is revolved about x-axis.

Sol: $\sqrt{25-x^2}=3 \Leftrightarrow 25-x^2=9$

$\Leftrightarrow x^2=16 \Rightarrow x=\pm 4$

$\Rightarrow \int_{-4}^{4} (16-x^2)\,dx$ = answer.

Short ans:

① Evaluate $A=\int x^2 e^{2x}\,dx$.

Use integral by part: $\int u\,dv = uv - \int v\,du$

$u=x^2,\ du=2x\,dx$

$v=\int e^{2x}dx = \frac{1}{2}e^{2x}$

$A=\frac{1}{2}x^2 e^{2x} - \int \frac{1}{2}\cdot e^{2x}\cdot 2x\,dx$

$A=\frac{1}{2}x^2e^{2x} - \underbrace{\int x e^{2x}\,dx}_{B}$

Find B:

$u=x,\ du=dx$

$v=\int e^{2x}dx=\frac{1}{2}e^{2x}$

$B=\frac{1}{2}xe^{2x}-\int\frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}$

$\Rightarrow A=\frac{1}{2}x^2e^{2x}-\left(\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right)$

3) Find the second derivative $\frac{\partial^2 f}{\partial x\partial y}$ of

$f(x,y)=xe^y-\frac{1}{y}\cos\left(\frac{x}{y}\right)+\ln(x^3+y^2)$.

• Note: we need to find $\frac{\partial f}{\partial x}$ first, then find $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$.

Note: $(\ln x)'=\frac{1}{x}$

$\ln(u)'=\frac{u'}{u}$

$(\cos u)'=-u'\sin$

$(\sin u)'=u'\cos u$

• $\frac{\partial f}{\partial x}=(xe^y)'-\left(\frac{1}{y}\cdot\cos\frac{x}{y}\right)'+\ln(x^3+y^2)'$

Now treat x is variable, y is constant.

$\frac{\partial f}{\partial x}=\left[(x)'.(e^y)+(x)(e^y)'\right]-\left[\left(\frac{1}{y}\right)'\cos\frac{x}{y}+\left(\cos\frac{x}{y}\right)'.\frac{1}{y}\right]+\left[\frac{(x^3+y^2)'}{x^3+y^2}\right]$

y is constant

$\frac{\partial f}{\partial x}=\left[e^y+0\right]-\left[0-\frac{1}{y}\sin\left(\frac{x}{y}\right)\right]+\left[\frac{3x^2}{x^3+y^2}\right]$

$\frac{\partial f}{\partial x}=e^y+\frac{1}{y}\sin\left(\frac{x}{y}\right)+\frac{3x^2}{x^3+y^2}$

Now: $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, so y is variable now, x is constant.

$\Rightarrow \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left[e^y+\frac{1}{y}\cdot\sin\left(\frac{x}{y}\right)+\frac{3x^2}{x^3+y^2}\right]$

$\frac{\partial^2 f}{\partial x\partial y}=(e^y)'+\left[\left(\frac{1}{y}\right)'\cdot\sin\left(\frac{x}{y}\right)+\sin\left(\frac{x}{y}\right)'\left(\frac{1}{y}\right)\right]+\left[\frac{0-2y.3x^2}{(x^3+y^2)^2}\right]$

$\frac{\partial^2 f}{\partial x\partial y}=e^y+\left[\frac{-1}{y^2}\sin\left(\frac{x}{y}\right)+\left(\frac{x}{y}\right)'\cos\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right)\right]-\frac{2y.3x^2}{(x^3+y^2)^2}$

$\frac{\partial^2 f}{\partial x\partial y}=e^y-\frac{1}{y^2}\sin\left(\frac{x}{y}\right)-\frac{x}{y^3}\cos\left(\frac{x}{y}\right)-\frac{6yx^2}{(x^3+y^2)^2}$

Make sure you check my work, maybe I made some mistakes.

Short:

Find the area of the region bounded by the graph of $y = x^3 + x^2 + 1$ and $y = 2x^2 + 2x + 1$

Sol: Intersection point:

$$x^3 + x^2 + 1 = 2x^2 + 2x + 1 \Leftrightarrow x^3 - x^2 - 2x = 0$$

$$\Leftrightarrow x(x^2 - x - 2) = 0 \Leftrightarrow x(x-2)(x+1) = 0$$

$$\Rightarrow x = -1; \quad x = 0; \quad x = 2.$$

$$A = \left|\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx\right| + \left|\int_0^1 (x^3 - x^2 - 2x)\right| = \cdots = \left|-\frac{8}{3}\right| + \frac{5}{12} = \frac{37}{12}.$$

(we do not know which graph on top so take absolute value to ensure get positive value of each area).