University of Ottawa

Department of Mathematics and Statistics

MAT 1302A : Mathematical Methods II

Professor: Hadi Salmasian

Third Midterm Exam – Version A

April 1, 2016

Surname ________________________ First Name ________________________

Student # ______________________ DGD ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(c) Write your student number at the top of each page in the space provided.
(d) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(e) You are strongly recommended to write in **pen**, not pencil.
(f) You may use the last page of the exam as scrap paper.
(g) Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. **Phones and devices must be turned off and put away in your bag**. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature ________________________

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	3	5	7	3	2	23
Grade							

1. [**3 points**] For each of the following sets, write **Yes** if the set is a subspace of $\mathbb{R}^n$ for the *given* value of n, and write **No** if it is not. You will receive .5 points for each correct answer and lose .25 points for each incorrect answer.

Yes The set $\left\{ \begin{bmatrix} x+y \\ y+2z \\ z+3x \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$, $n = 3$.

No The set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $n = 3$.

Yes A line in $\mathbb{R}^3$ which passes through the origin, $n = 3$.

Yes $\operatorname{Nul} B$ where B is a 6×9 matrix, $n = 9$.

No $\operatorname{Col} A$ where A is a 4×5 matrix, $n = 5$.

No $\operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, where $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^3$, $n = 4$.

2. [**3 points**] For each of the following statements, indicate if it is true (**T**) or false (**F**). You will receive .5 points for each correct answer, and will lose .25 points for each incorrect answer.

T If three vectors in $\mathbb{R}^3$ are linearly independent, then they form a basis for $\mathbb{R}^3$.

F $\mathbb{R}^4$ has a basis which consists of 5 vectors.

T If the rank of a matrix A of size 4×6 is equal to 3, then $\dim \operatorname{Nul} A = 3$

F If A is an invertible matrix then $\det A = 0$.

T If A is a square matrix, then $\det A = \det A^T$.

T There exists an imaginary number z such that $z^2 = -2$.

3. [**5 points**] Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{bmatrix}$$

Solution: The best method for such a large matrix is the row reduction method:

$$\begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{bmatrix} \xrightarrow{\substack{L_3 \to L_3 + 2L_1 \\ L_4 \to L_4 - 3L_1 \\ L_5 \to L_5 - 3L_1}} \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{bmatrix}$$

$$\xrightarrow{\substack{L_4 \to L_4 + L_2 \\ L_5 \to L_5 + 2L_2}} \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is triangular and therefore its determinant is the product of diagonal entries, i.e., -24. Since we used one interchange throughout, the determinant of A is equal to $(-1)(-24) = 24$.

4. Let

$$A = \begin{bmatrix} 1 & 2 & -4 & 4 & 6 \\ 5 & 1 & -9 & 2 & 10 \\ 4 & 6 & -9 & 12 & 15 \\ 3 & 4 & -5 & 8 & 9 \end{bmatrix}.$$

We are given that the following matrix is an echelon form of A:

$$\begin{bmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) [**2 points**] Find a basis for Col A.
(b) [**1 points**] Determine the rank of A.
(c) [**3 points**] Find a basis for Nul A
(d) [**1 point**] Find dim Nul A.

Solution:

(a) By the given echelon form, the columns 1, 2 and 3 are pivot columns. We conclude that the same columns in A form a basis of Col A. That is, the basis is given by:

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ -5 \end{bmatrix} \right\}$$

(b) From (a) it follows that the rank of A is 3.

(c) The reduced echelon form of A is obtained from its echelon form as follows:

$$\begin{bmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{5}R_3} \begin{bmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 8R_3]{R_2 \to R_2 - 3R_3} \begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ 0 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The homogeneous linear system associated to the reduced echelon form is

$$\begin{cases} x_1 = 0 \\ x_2 + 2x_4 + x_5 = 0 \\ x_3 - x_5 = 0 \end{cases}$$

with x_4 and x_5 free variables. Then the vector parametric form of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

and the obtained basis of $\operatorname{Nul} A$ is:

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) $\dim \operatorname{Nul} A = 2$.

5. [**3 points**] Write the following complex numbers in the standard form $a + bi$, $a, b \in \mathbb{R}$.

(a) $(2+i)\overline{(3-2i)}$.

(b) $\dfrac{1+i}{1-i}$.

Solution: We have

$$(2+i)\overline{(3-2i)} = (2+i)(3+2i) = 6 + 4i + 3i + (i)(2i) = 6 + 7i - 2 = 4 + 7i.$$

and

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i+i^2}{1^2+1^2} = \frac{2i}{2} = i.$$

6. [**2 points**] Suppose that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 8.$$

Calculate

$$\begin{vmatrix} a_1 - c_1 & a_2 - c_2 & a_3 - c_3 \\ c_1 & c_2 & c_3 \\ 3b_1 & 3b_2 & 3b_3 \end{vmatrix}$$

Solution: We have

$$\begin{vmatrix} a_1 - c_1 & a_2 - c_2 & a_3 - c_3 \\ c_1 & c_2 & c_3 \\ 3b_1 & 3b_2 & 3b_3 \end{vmatrix} = 3 \begin{vmatrix} a_1 - c_1 & a_2 - c_2 & a_3 - c_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= -3 \begin{vmatrix} a_1 - c_1 & a_2 - c_2 & a_3 - c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -24$$