COMP 233, Winter 2016 **CONCORDIA UNIVERSITY**

PROBABILITY AND STATISTICS FOR COMPUTER SCIENCE.
Assignment 1. Solutions.

1. (6 points) Suppose the sample space § consists of all seven-letter words having distinct alphabetic characters.

   (a) How many words are there in § ?

   (b) How many "special" words are in § for which only the second, the fourth, and the sixth characters are vowels, *i.e.*, one of $\{a, e, i, o, u, y\}$?

   (c) Assuming the outcomes in § to be equally likely, what is the probability of drawing such a special word?

   SOLUTION:

   (a) $\frac{26!}{(26-7)!} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 =$ 3,315,312,000.

   (b) $6 \cdot 5 \cdot 4 \cdot 20 \cdot 19 \cdot 18 \cdot 17 =$ 1,3953,600.

   (c) $\frac{13953600}{3315312000} \approx 0.42\%$.

2. (4 points) If $P(E) = 0.9$ and $P(F) = 0.9$, show that $P(EF) \geq 0.8$. In general, prove Bonferroni's inequality, namely that

$$P(EF) \geq P(E) + P(F) - 1.$$

   SOLUTION:

$$1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$$

   implies Bonferroni's inequality

$$P(EF) \geq P(E) + P(F) - 1.$$

   Using Bonferroni's inequality with the values provided we get

$$P(EF) \geq P(E) + P(F) - 1 = 0.9 + 0.9 - 1 = 0.8.$$

3. (6 points) Three balls are selected at random from a bag containing 2 red, 3 green, and 4 blue balls.

   (a) What would be an appropriate sample space § ?

   (b) What is the the number of outcomes in § ?

   (c) What is the probability that all three balls are red?

   (d) What is the probability that all three balls are green?

   (e) What is the probability that all three balls are blue?

   (f) What is the probability of one red, one green, and one blue ball?

SOLUTION:

(a) The set of all subsets of three elements from $B = \{r_1, r_2, g_1, g_2, g_3, b_1, b_2, b_3, b_4\}$.

(b) $\binom{9}{3} = 84$. (c) Zero ! (d) $1/84$. (e) $\binom{4}{3} / \binom{9}{3} = 4/84 = 1/21$.

(f) $\binom{2}{1}\binom{3}{1}\binom{4}{1} / \binom{9}{3} = (2 \cdot 3 \cdot 4)/84 = 2/7$.

4. (4 points) An urn contains four balls, labeled 1 to 4. Balls are drawn at random one by one, without replacement, until the sum of the numbers on the balls drawn exceeds 4. The sequence of balls drawn is noted.

   (a) (2 points) Write down the sample space for this experiment.

   (b) Let E be the event "one of the balls drawn is 1", and F the event "the final sum of numbers on the balls drawn is even". Give the set of outcomes corresponding to each of the following events

      (i) (1 point) "both E and F occur"

      (ii) (1 point) "neither E nor F occurs"

SOLUTION:

(a) The sample space S contains exactly the following outcomes: $(1,2,3)$, $(1,2,4)$, $(1,3,2)$, $(1,3,4)$, $(1,4)$, $(2,1,3)$, $(2,1,4)$, $(2,3)$, $(2,4)$, $(3,1,2)$, $(3,1,4)$, $(3,2)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$.

(b) The event E, which is defined to be "one of the balls drawn is 1", when expressed as a subset of the sample space is

$$E = \{(1,2,3),(1,2,4),(1,3,2),(1,3,4),(1,4),(2,1,3),(2,1,4),(3,1,2),(3,1,4),(4,1)\}.$$

The event F, *i.e.*, "the final sum of numbers on the balls drawn is even", is the set

$$F = \{(1,2,3),(1,3,2),(1,3,4),(2,1,3),(2,4),(3,1,2),(3,1,4),(4,2)\}.$$

(i) The event "both E and F occur" is the set

$$E \cap F = \{(1,2,3),(1,3,2),(1,3,4),(2,1,3),(3,1,2),(3,1,4)\}.$$

(ii) The event "neither E nor F occurs" is the set

$$E^c \cap F^c = (E \cup F)^c = \{(2,3),(3,2),(3,4),(4,3)\}.$$

5. (4 points) How many integer solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 17 ,$$

if we require that

$$x_1 \geq 1 \quad , \quad x_2 \geq 2 \quad , \quad x_3 \geq 3 \ ?$$

SOLUTION: With a slack variable $\tilde{x}_4$, and letting $x_1 = \tilde{x}_1 + 1$, $x_2 = \tilde{x}_2 + 2$, $x_3 = \tilde{x}_3 + 3$, we have $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{x}_4 = 11$, with all $\tilde{x}_i \geq 0$. The number of solutions is then seen to be $\binom{11+3}{3} = 364$.

6. (4 points) What is the probability that the sum is less than or equal to 9 in three rolls of a die?

   SOLUTION: The number of such sequences is the number of integer solutions of the inequality $x_1 + x_2 + x_3 \leq 9$, with $1 \leq x_1, x_2, x_3 \leq 6$, Let $x_1 = \tilde{x}_1 + 1$, $x_2 = \tilde{x}_2 + 1$, $x_3 = \tilde{x}_3 + 1$, and introduce a slack variable $\tilde{x}_4$. Then the problem becomes: How many nonnegative integer solutions are there to the equation $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{x}_4 = 6$, with $0 \leq \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \leq 5$. The unrestricted problem has has $\binom{9}{3} = 84$ solutions, from which we must subtract the 3 impossible solutions $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (6, 0, 0.0)$, $(0, 6, 0, 0)$, $(0, 0, 6, 0)$. Thus the probability that the sum of 3 rolls is less that or equal to 9 is $\frac{84-3}{6^3} = \frac{81}{216} = 0.375$.

7. (6 points) Rooks (or castles) are chess pieces that are only allowed to move horizontally or vertically on a chessboard. Assume that rooks must be placed on one of the 64 allowed positions on the board, and no two rooks can share the same position.

   How many ways are there of placing 8 indistinguishable rooks on a standard 8×8 chessboard:

   (a) if there are no restrictions on their placement, other than those stated above?

   (b) if no rook can be in a position that attacks another?

   SOLUTION:

   (a) The number of unrestricted placements of 8 indistinguishable rooks on a standard chessboard is equal to the number of ways of picking 8 squares on the chessboard to place the rooks. This number is simply $\binom{64}{8}$.

   (b) The restriction here is that no two rooks can be in the same row or column of the board. We do the count for distinguishable rooks, an then divide by 8! to get the count for the indistinguishable case.

   The number of ways of placing rook 1 is 64, but once this rook has been placed, say at position (i, j), no further rooks can be placed in the ith row and jth column. Thus there are 49 $(= 7^2)$ squares left to place rook 2. Eliminating the rows and columns containing rooks 1 and 2 leaves 36 $(= 6^2)$ squares for rook 3, and so on until rook 8, which can only be placed on the one uncovered square remaining. So, the number of non-attacking placements of 8 distinguishable rooks is $8^2 \times 7^2 \times 6^2 \times \cdots \times 1^2 = (8!)^2$. Dividing this by 8!, we find the number of non-attacking placements of 8 indistinguishable rooks to be 8!.

   Here is another way of doing this count. Since there are 8 columns and 8 rooks to be placed, and there can be at most one rook in each column, we see that each column must in fact have exactly one rook. The number of positions a rook can be place in the 1st column is 8; after this the the number of positions to place a rook in the 2nd column is 7 (it cannot be placed in the same row as the rook in column 1); and so on, all the way to the 8th column. Thus, the total number of placements of 8 non-attacking rooks is $8 \times 7 \times \cdots \times 1 = 8!$.

8. (6 points) A child has 12 blocks: 6 black, 4 red, 1 white, and 1 yellow.

   (a) If the child puts the blocks in a line, how many different arrangements are possible?

   (b) If one of the arrangements in part (a) is randomly selected, what is the probability that no two black blocks are next to each other?

   SOLUTION:

   (a) The number of ways the blocks can be arranged in a line is the number of distinguishable permutations of 12 objects of 4 different types such that 6 are type 1 (black), 4 are type 2 (red), 1 is type 3 (white), and 1 is type 4 (yellow). The number of all such distinguishable permutations is

   $$\frac{12!}{6!4!1!1!} = 27720.$$

   (b) Since there are 12 blocks, six of which are black, the condition that no two black blocks are neighbors means that between every two consecutive black blocks there is at least one block of a different color. The line either starts with a black block or with a block of different color. If the line starts with a different color, then between two black blocks there must be exactly one block of different color. If the line starts with a black block, then either black and different colors alternate (and the line ends with a different color), or there exist exactly two black blocks between which there are exactly two blocks of different color, and in the rest of the line the black and differently colored blocks alternate (the line ends with a black block). In the latter case, there are 5 ways of choosing where to insert the two blocks of different color between the two black blocks. Therefore, if we consider the non-black blocks indistinguishable, there are 7 different arrangements such that no two black blocks are next to each other. In all 7 cases, the 6 non-black blocks (now viewed as 4 red, 1 white, and 1 yellow) can be arranged in

   $$\frac{6!}{4!1!1!} = 30$$

   different ways. Thus the number of arrangements in question is $7 \cdot 30 = 210$ and the desired probability is

   $$\frac{210}{27720} \approx 0.00757.$$

9. (4 points) A box contains three coins, one of which is fair, one double-headed (*i.e.*, heads on both sides), and the third is biased in such a way that it comes up heads with probability 3/4. A coin is drawn at random from the box and flipped twice. If both flips result in heads, what is the probability that the coin drawn was double-headed?

   SOLUTION:

   Let F, D and B be the events "fair coin selected", "double-headed coin selected" and "biased coin selected", respectively. Also, let HH be the event "both flips of the selected coin result in heads". We are asked to determine $P(D|HH)$.

We are given that $P(F) = P(D) = P(B) = 1/3$, and (assuming independence)

$$P(HH|F) = (1/2)^2 = 1/4, \quad P(HH|D) = 1, \quad P(HH|B) = (3/4)^2 = 9/16.$$

So, applying Bayes' rule, we find

$$\begin{aligned} P(D|HH) &= \frac{P(HH|D)P(D)}{P(HH|F)P(F) + P(HH|D)P(D) + P(HH|B)P(B)} \\ &= \frac{1 \times 1/3}{(1/4 \times 1/3) + (1 \times 1/3) + (9/16 \times 1/3)} = \frac{16}{29}. \end{aligned}$$

10. (4 points) A monkey at a typewriter types each of the 26 capital letters of the alphabet exactly once, the order being random.

    (a) (2 points) What is the probability that the word "MONKEY" appears somewhere in the string of letters?

    (b) (2 points) How many independent monkey typists are needed so that the probability that the word "MONKEY" appears at least once is greater than 0.9?

SOLUTION:

(a) The number of permutations of the 26 letters of the alphabet is 26!. The number of permutations in which "MONKEY" appears as a word is 21!, namely, the number of ways to place the word "MONKEY' (21), times the number of ways to place the remaining 20 letters (20!). Thus the probability that the word "MONKEY" appears in a random permutation of the 26 letters of the English alphabet is $21!/26! = 1/7893600 \approx 1.267 \times 10^{-7}$.

(b) Note that by the result of part (a), the probability that the word "MONKEY" does not appear in a random permutation of the letters of the alphabet is $1 - 1/7893600 \approx 0.99999987$.

So, if we have n independent monkeys each typing out a random permutation of the 26 letters of the alphabet, the probability that none produces the string "MONKEY" is $(1 - 1/7893600)^n$.

Thus the probability that at least one produces the string "MONKEY" is

$$1 - (1 - 1/7893600)^n.$$

We need to determine the values of n that would satisfy $1-(1-1/7893600)^n > 0.9$, or equivalently, $(1 - 1/7893600)^n < 0.1$. Taking base-10 logarithms, we obtain

$$n \log_{10}(1 - 1/7893600) < -1$$

and since $\log_{10}(1 - 1/7893600) \approx -5.50185 \times 10^{-8}$, we see that

$$n > \frac{-1}{-5.50185 \times 10^{-8}} \approx 18175685.$$

So, we need more than 18,175,685 monkeys typing independently at random so that, with probability greater than 0.9, at least one of them types the word "MONKEY".