Part 1: Understanding Questions

1) What is a vector space?
2) What is a subspace?
3) What is the subspace test?
4) What is the space of column vectors R^n or C^n?
5) What is the space of polynomials P_k?
6) What is the space of matrices $R^{m\times n}$?

Part 2: Practice the Concepts using the following exercises:

1) Consider a space of 2x2 matrices with real entries where we redefine addition and scalar multiplication as follows:

$$\begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{b_1} \\ \boldsymbol{c_1} & \boldsymbol{d_1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{a_2} & \boldsymbol{b_2} \\ \boldsymbol{c_2} & \boldsymbol{d_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{b_2} \\ \boldsymbol{c_2} & \boldsymbol{d_1} \end{bmatrix} \qquad \boldsymbol{k}\begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{b_1} \\ \boldsymbol{c_1} & \boldsymbol{d_1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{k} & \boldsymbol{k} \\ \boldsymbol{k} & \boldsymbol{k} \end{bmatrix}$$

   i) Explain what the addition and scalar multiplication are doing in your own words.
   ii) Which of the following vector space properties does it pass and which does it fail?
   ($x, y, z \in V$, $\alpha, \beta \in \boldsymbol{R}$)
   a) $x + (y + z) = (x + y) + z$
   b) $x + y = y + x$
   c) There is a zero vector $0_V \in V$ such that $0_V + x = x$ for every $x \in V$
   d) For every $x \in V$ we have $x' \in V$ such that $x + x' = 0_V$
   e) $\alpha(x + y) = \alpha x + \alpha y$
   f) $(\alpha + \beta)x = \alpha x + \beta x$
   g) $(\alpha\beta)x = \alpha(\beta x)$
   h) $1x = x$ for all $x \in V$ where 1 is the multiplicative identity in $\boldsymbol{R}$**.**
   iii) Is this space a vector space under this addition and scalar multiplication?

2) Consider a space of quadratic polynomials with real coefficients where we redefine addition and scalar multiplication as follows:

$$(\boldsymbol{a_1x^2 + b_1x + c_1}) + (\boldsymbol{a_2x^2 + b_2x + c}) = \boldsymbol{a_1a_2x^2 + b_1b_2x + c_1c_2}$$
$$\boldsymbol{k(a_1x^2 + b_1x + c_1) = 0x^2 + 0x + 0}$$

   i) Explain what the addition and scalar multiplication are doing in your own words.
   ii) Which of the following vector space properties does it pass and which does it fail?
   ($x, y, z \in V$, $\alpha, \beta \in \boldsymbol{R}$)
   i) $x + (y + z) = (x + y) + z$
   j) $x + y = y + x$
   k) There is a zero vector $0_V \in V$ such that $0_V + x = x$ for every $x \in V$
   l) For every $x \in V$ we have $x' \in V$ such that $x + x' = 0_V$
   m) $\alpha(x + y) = \alpha x + \alpha y$
   n) $(\alpha + \beta)x = \alpha x + \beta x$
   o) $(\alpha\beta)x = \alpha(\beta x)$
   p) $1x = x$ for all $x \in V$ where 1 is the multiplicative identity in $\boldsymbol{R}$**.**
   iii) Is this space a vector space under this addition and scalar multiplication?

3) Determine which of the following are subspaces of a known vector space using the subspace test:

   i) $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + b + c = 0\right\}$ ii) $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + b = 5\right\}$ iii) $\{0x^2 + 0x + 0\}$

   iv) $\{ax^2 + b \mid a + b = 0\}$ v) $\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid ab = 0\right\}$ vi) $\left\{\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a - b = 0\right\}$

Part 3: Thinking Questions

1) Is the following set a subspace of P_2?
   The set of all quadratics that have real solutions
2) Prove that 2x2 matrices with entries under $GF(2)$ and scalars under $GF(2)$ is a vector space under addition and scalar multiplication.