Assignment 2 Solutions

1. For each of the arguments below, formalize them in propositional logic. If the argument is valid identify which inference rule was used. If the argument is invalid, state whether the inverse or converse error was made.

   (a) All healthy people eat an apple a day.
   Helen eats an apple a day.
   $\therefore$ Helen is a healthy person.

   Solution:

   Invalid. The argument is of the form

   Healthy(Helen) $\rightarrow$ EatApple(Helen).
   EatApple(Helen)
   $\therefore$ Healthy Helen

   It is an example of the converse error.

   (b) All healthy people eat an apple a day.
   Herbert is not a healthy person.
   $\therefore$ Herbert does not eat an apple a day.

   Solution:

   Invalid. The argument is of the form

   Healthy(Herbert) $\rightarrow$ EatApple(Herbert).
   $\neg$ Heathy(Herbert)
   $\therefore \neg$ EatApple(Herbert)

   It is an example of the inverse error.

   (c) If a product of two numbers is 0, then at least one of the numbers is 0.
   For a particular number x, neither $(x-1)$ nor $(x+1)$ equals 0.
   $\therefore$ The product $(x-1)(x+1)$ is not 0.

   Solution:

   Valid. The argument is Universal Modus Tollens.

2. For each of the following, determine whether argument is valid. You may use a counterexample or equivalence transformations to justify your answer.

   (a) $p \to q$
       $\underline{\neg p \quad\quad}$
       $\therefore \neg q$

       Solution: Invalid. Counterexample: p is *False* and q is *True*.

   (b) $p \to r$
       $q \to r$
       $\underline{\neg(p \vee q)}$
       $\therefore \neg r$

       Solution: Invalid. Counterexample: p and q are *False* and r is *True*.

   (c) $p \to r$
       $q \to r$
       $\underline{q \vee \neg r}$
       $\therefore \neg p$

       Solution: Invalid. Counterexample: p, q and r are *True*.

3. For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step. For examples, see pages 73 and 74 in textbook.

(a) Premise: $\{\neg p \vee q \rightarrow r,\ s \vee \neg q,\ \neg t,\ p \rightarrow t,\ \neg p \wedge r \rightarrow \neg s\}$, conclusion: $\neg q$.

Solution:

Step	Conclusion	Reason
1.	$p \rightarrow t$	Premise
2.	$\neg t$	Premise
3.	$\neg p$	Modus Tollens using (1) and (2)
4.	$\neg p \vee q$	Addition using (3)
5.	$\neg p \vee q \rightarrow r$	Premise
6.	r	Modus Ponens using (4) and (5)
7.	$\neg p \wedge r$	Conjunction using (3) and (6)
8.	$\neg p \wedge r \rightarrow \neg s$	Premise
9.	$\neg s$	Modus Ponens using (7) and (8)
10.	$s \vee \neg q$	Premise
11.	$\neg q$	Disjunctive Syllogism using (9) and (10)

(b) Premise: $\{\neg p \rightarrow r \wedge \neg s,\ t \rightarrow s,\ u \rightarrow \neg p,\ \neg w,\ u \vee w\}$, conclusion: $\neg t \vee w$.

Solution:

Step	Conclusion	Reason
1.	$u \vee w$	Premise
2.	$\neg w$	Premise
3.	u	Disjunctive Syllogism using (1) and (2)
4.	$u \rightarrow \neg p$	Premise
5.	$\neg p$	Modus Ponens using (3) and (4)
6.	$\neg p \rightarrow r \wedge \neg s$	Premise
7.	$r \wedge \neg s$	Modus Ponens using (5) and (6)
8.	$\neg s$	Simplification using (7)
9.	$t \rightarrow s$	Premise
10.	$\neg t$	Modus Tollens using (8) and (9)
11.	$\neg t \vee w$	Addition using (10)

(c) Premise: $\{p \vee q,\ q \to r,\ p \wedge s \to t,\ \neg r,\ \neg q \to u \wedge s\}$, conclusion: t.

Solution:

Step	Conclusion	Reason
1.	$\neg r$	Premise
2.	$q \to r$	Premise
3.	$\neg q$	Modus Tollens using (1) and (2)
4.	$\neg q \to u \wedge s$	Premise
5.	$u \wedge s$	Modus Ponens using (3) and (4)
6.	s	Simplification using (5)
7.	$p \vee q$	Premise
8.	p	Disjunctive Syllogism using (3) and (7)
9.	$p \wedge s$	Conjunction using (6) and (8)
10.	$p \wedge s \to t$	Premise
11.	t	Modus Ponens using (9) and (10)

4. Use rules of inference to show that

(a) $\forall x \left(R(x) \to \left(S(x) \vee Q(x)\right)\right)$
$\exists x \left(\neg S(x)\right)$

$\therefore \exists x \left(R(x) \to Q(x)\right)$

Solution:

(1)	$\exists x \left(\neg S(x)\right)$	Premise
(2)	$\neg S(c)$	Existential instantiation from (1)
(3)	$\forall x \left(R(x) \to \left(S(x) \vee Q(x)\right)\right)$	Premise
(4)	$R(c) \to \left(S(c) \vee Q(c)\right)$	Universal instantiation from (3)
(5)	$\neg R(c) \vee \left(S(c) \vee Q(c)\right)$	$p \to q \equiv \neg p \vee q$ from (4)
(6)	$\left(\neg R(c) \vee S(c)\right) \vee Q(c)$	Associativity from (5)
(7)	$\left(S(c) \vee \neg R(c)\right) \vee Q(c)$	Commutativity from (6)
(8)	$S(c) \vee \left(\neg R(c) \vee Q(c)\right)$	Associativity from (7)
(9)	$\neg R(c) \vee Q(c)$	Disjunctive syllogism from (8) and (2)
(10)	$R(c) \to Q(c)$	$p \to q \equiv \neg p \vee q$ f rom (9)
(11)	$\exists x \left(R(x) \to Q(x)\right)$	Existential generalization from (10)

(b) $\forall x \Big(P(x) \vee Q(x) \Big)$
$\forall x \Big(\big(\neg P(x) \wedge Q(x) \big) \rightarrow R(x) \Big)$

$\therefore \forall x \Big(\neg R(x) \rightarrow P(x) \Big)$

Solution:

(1)	$\forall x \Big(P(x) \vee Q(x) \Big)$	Premise
(2)	$P(c) \vee Q(c)$	Universal Instantiation from (1)
(3)	$\forall x \Big(\big(\neg P(x) \wedge Q(x) \big) \rightarrow R(x) \Big)$	Premise
(4)	$\Big(\neg P(c) \wedge Q(c) \Big) \rightarrow R(c)$	Universal Instantiation from (3)
(5)	$\neg \Big(\neg P(c) \wedge Q(c) \Big) \vee R(c)$	$p \rightarrow q \equiv \neg p \vee q$ from (4)
(6)	$\Big(\neg\neg P(c) \vee \neg Q(c) \Big) \vee R(c)$	De Morgan from (5)
(6)	$\Big(P(c) \vee \neg Q(c) \Big) \vee R(c)$	Double negation from (6)
(7)	$\Big(P(c) \vee R(c) \Big) \vee \neg Q(c)$	Commutativity and Associativity fro m (6)
(8)	$P(c) \vee \Big(P(c) \vee R(c) \Big)$	Resolution from (2) and (7)
(9)	$\Big(P(c) \vee P(c) \Big) \vee R(c)$	Associativity from (8)
(10)	$P(c) \vee R(c)$	Idempotency from (9)
(11)	$R(c) \vee P(c)$	Commutativity from (10)
(12)	$\neg R(c) \rightarrow P(c)$	$p \rightarrow q \equiv \neg p \vee q$ from (11)
(13)	$\forall x \Big(\neg R(x) \rightarrow P(x) \Big)$	Universal generalization from (11)

(c) $\forall x \Big(P(x) \land Q(x) \Big)$

$\therefore \Big(\forall x P(x) \Big) \land \Big(\forall x Q(x) \Big)$

Solution:

(1)	$\forall x \Big(P(x) \land Q(x) \Big)$	Premise
(2)	$P(c) \land Q(c)$	Universal Instantiation from (1)
(3)	$P(c)$	Simplification from (2)
(4)	$\forall x P(x)$	Universal Generalization from (3) from (3)
(5)	$Q(c)$	Simplification from (2)
(6)	$\forall x Q(x)$	Universal Generalization from (5)
(6)	$\Big(\forall x P(x) \Big) \land \Big(\forall x Q(x) \Big)$	Conjunction from (4) and (6)

5. Prove that the following four statements are equivalent:

   (a) n^2 is odd.

   (b) $1 - n$ is even.

   (c) n^3 is odd.

   (d) $n^2 + 1$ is even.

Solution:

We will show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$. The claim then follows (why?).

- $(a) \Rightarrow (b)$. We show the contrapositive $odd(1-n) \Rightarrow even(n^2)$.

  $odd(1-n) \Rightarrow 1 - n = 2k+1$, for some $k \in \mathbb{Z}$
  $\Rightarrow n = -2k$
  $\Rightarrow n = 2(-k)$
  $\Rightarrow n^2 = \Big(2(-k)\Big)^2 = 2\Big(2k^2\Big)$
  $\Rightarrow even(n^2)$.

- $(b) \Rightarrow (c)$.
  $even(1-n) \Rightarrow 1 - n = 2k$, for some $k \in \mathbb{Z}$
  $\Rightarrow -n = 2k - 1$
  $\Rightarrow n = -2k + 1$
  $\Rightarrow n^3 = \Big(-2k+1\Big)^3$
  $\Rightarrow n^3 = -8k^3 + 12k^2 - 6k + 1$
  $\Rightarrow n^3 = 2\Big(-4k^3 + 6k^2 - 3k\Big) + 1$
  $\Rightarrow odd(n^3)$.

- $(c) \Rightarrow (d)$. We show the contrapositive $odd(n^2+1) \Rightarrow even(n^3)$.
  $odd(n^2+1) \Rightarrow even(n^2)$
  $\Rightarrow n^2 = 2k$, for some $k \in \mathbb{Z}$
  $\Rightarrow n^3 = n \cdot n^2 = n(2k) = 2(nk)$
  $\Rightarrow even(n^3)$.

- $(d) \Rightarrow (a)$.
  $even(n^2+1) \Rightarrow n^2 + 1 = 2k$, for some $k \in \mathbb{Z}$
  $\Rightarrow n^2 = 2k - 1 = 2k - 1 + 1 - 1$
  $\Rightarrow n^2 = 2k - 2 + 1 = 2(k-1) + 1$
  $\Rightarrow odd(n^2)$.

6. (a) Give a direct proof of: "If x is an odd integer and y is an even integer, then $x + y$ is odd."

Solution:

$odd(x) \Rightarrow x = 2k + 1$, for some $k \in \mathbb{Z}$
$even(y) \Rightarrow y = 2k'$, for some $k' \in \mathbb{Z}$
$\Rightarrow x + y = 2k + 1 + 2k' = 2(k + k') + 1$
$\Rightarrow odd(x + y)$.

(b) Give a proof by contradiction of: "If n is an odd integer, then n^2 is odd."

Solution: Suppose to the contrary that $odd(n)$ and $even(n^2)$.

$odd(n) \Rightarrow n = 2k + 1$, for some $k \in \mathbb{Z}$
$\Rightarrow n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
$\Rightarrow odd(n^2)$, which contradicts $even(n^2)$.

(c) Give a proof by contraposition of: "If n is an odd integer, then $n + 2$ is odd."

Solution: Using a proof by contrapositive, we need to prove $even(n+2) \Rightarrow even(n)$.

$even(n + 2) \Rightarrow n + 2 = 2k$, for some $k \in \mathbb{Z}$
$\Rightarrow n = 2k - 2 = 2(k - 1) \Rightarrow even(n)$.

7. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, e.g., by giving a counterexample.

   (a) For all positive $x, y \in \mathbb{R}$, if x is irrational and y is irrational then $x+y$ is irrational.

   Solution:

   False: Counterexample: Let $x_1 = \frac{7}{4} - \sqrt{2}$ and $x_2 = \frac{7}{4} + \sqrt{2}$. Then both x_1 and x_2 are positive and irrational, while their sum is $\frac{7}{2}$, which is rational.

   For completeness we must also demonstrate that x_1 and x_2 are indeed irrational. This is most easily done by contradiction: Suppose $x_1 = \frac{7}{4} - \sqrt{2}$ is rational. Then $x_1 = \frac{7}{4} - \sqrt{2} = \frac{p}{q}$, for positive integers p and q. It follows that $\sqrt{2} = \frac{7}{4} - \frac{p}{q} = \frac{7q-4p}{4q}$, which is rational. But $\sqrt{2}$ is known to be irrational, and thus we have a contradiction. The proof that x_2 is irrational is very similar to that for x_1. (The fact that $\sqrt{2}$ is irrational is demonstrated in the textbook.)

   (b) For all positive $x, y \in \mathbb{R}$, if x is irrational and y is rational then xy is irrational.

   Solution:

   True: assuming $x \neq 0$ and $y \neq 0$. We do a proof by contradiction.

   Counterassumption: Suppose x is irrational, y is rational, and xy is rational.

   Since y is rational, $y = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, and $b \neq 0$.

   Since xy is rational, $xy = \frac{c}{d}$, where $c, d \in \mathbb{Z}$, and $d \neq 0$.

   Hence $x = \frac{xy}{y} = \frac{\frac{c}{d}}{\frac{a}{b}} = \frac{cb}{ad}$.

   We have $d \neq 0$, and since $y \neq 0$ we have $a \neq 0$. Consequently x is rational; a contradiction to our counterassumption. Therefore xy must be irrational.

   (c) $\sqrt{3}$ is irrational.

   Solution:

   True: We prove this by contradiction: Suppose that $\sqrt{3}$ is rational. We may also assume that $\sqrt{3}$ is positive, that is, we consider only the positive root.

   Then $\sqrt{3} = \frac{p}{q}$, for positive integers p and q. By factoring out common factors we may assume that the only common divisor of p and q is 1. Now from $\sqrt{3} = \frac{p}{q}$ it follows that $p^2 = 3q^2$. Thus $3|p^2$. It is easily seen that therefore $3|p$ (see below). Thus $p = 3k$ for some positive integer k. Hence $p^2 = (3k)^2 = 3q^2$, from which it follows that $3k^2 = q^2$. Thus also $3|q^2$. Again this implies that $3|q$. Therefore we have found that both p and q are divisible by 3, which contradicts the fact that their only common divisor is 1.

   For completeness we prove a fact that was used two times in the above proof, namely that $3|p^2$ implies $3|p$. This is most easily done by proving the contrapositive: If p is not divisible by 3 then p^2 is not divisible by 3. So assume p is not divisible by 3. Then p can be written as $p = 3k+1$ or $p = 3k+2$. In the first case this gives $p^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$, which is not divisible by 3. In the second case we have $p^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$, which is not divisible by 3 either.

8. Consider the statement concerning integers "*If* $m + n$ *is even, then* $m - n$ *is even.*"

(a) Give a direct proof of the statement.

Solution:
$even(m + n) \Rightarrow m + n = 2k$, for some $k \in \mathbb{Z}$
$\Rightarrow n = (2k - m)$
$\Rightarrow m - n = m - 2k + m = 2m - 2k = 2(m - k)$
$\Rightarrow even(m - n)$.

(b) Give a proof by contraposition of the statement.

Solution: We need to prove $odd(m - n) \Rightarrow odd(m + n)$.

$odd(m - n) \Rightarrow m - n = 2k + 1$, for some $k \in \mathbb{Z}$
$\Rightarrow m = 2k + 1 + n$
$\Rightarrow m + n = 2k + 1 + n + n = 2(k + n) + 1$
$\Rightarrow odd(m + n)$.

(c) Prove the statement by contradiction.

Solution: We need to prove that $even(m + n) \wedge odd(m - n)$ leads to a contradiction.

$odd(m - n) \Rightarrow m - n = 2k + 1$, for some $k \in \mathbb{Z}$
$\Rightarrow m = 2k + 1 + n$
$\Rightarrow m + n = 2k + 1 + n + n = 2(k + n) + 1$
$\Rightarrow odd(m + n)$, which contradicts $even(m + n)$.

Therefore, the given statement is true, that is, $even(m + n) \Rightarrow even(m - n)$.

Comment on Question 8 on the next page

Comment on Question 8: For the particular statement in question 8 the proof by contraposition and the proof by contradiction are essentially the same.

However, in a proof by contraposition of $P \Rightarrow Q$ we prove $\neg Q \Rightarrow \neg P$. In a proof by contradiction of $P \Rightarrow Q$ we assume P and $\neg Q$ and derive *some* contradiction. In question 8 (c) the contradiction happens to be $\neg P$.

Here is an example where the two proofs differ:

"*For all integers n, if $3n + 2$ is even then n is even.*"

- **Proof by contraposition:** We show $odd(n) \Rightarrow odd(3n + 2)$.

  $odd(n) \Rightarrow n = 2k + 1$
  $\Rightarrow 3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 4 + 1 = 2(3k + 2) + 1$
  $\Rightarrow odd(3n + 2)$.

- **Proof by contradiction:** Let $3n + 2$ be even and assume that n –contrary to the claim– is odd.

  Since $3n + 2$ is even, so is $3n$. We note that if we subtract an odd number from an even number, the result is odd. Since n is odd, $3n - n$ is odd. But $3n - n = 2n$ which is even. We have thus shown that our counter-assumption that n is odd leads to a contradiction.