MATH364 Winter 2016

Assignment 1

Due Wednesday January 20, 2016

1. Prove that $f : X \to Y$ is surjective if and only if for any subset $A \subset X$, we have

$$Y \setminus f(A) \subset f(X \setminus A).$$

When are the two last sets equal?

Solution: Assume that f is surjective, and let $y \in Y \setminus f(A)$, i.e. for all $x \in A$, $f(x) \neq y$. Since f is surjective, there is a $x \in X$ such that $f(x) = y$ and $x \notin A$, i.e. $x \in X \setminus A$, and $y \in f(X \setminus A)$.

For the converse, suppose now that f is not surjective. We have to show there exists a set A such that $Y \setminus f(A) \not\subset f(X \setminus A)$. Since f is not surjective, there is $y_0 \in Y \setminus f(X)$. Taking $A = X$, we have that $y_0 \notin f(X \setminus X) = f(\emptyset) = \emptyset$, and $Y \setminus f(X) \not\subset X \setminus X$.

The two sets are equal when f is bijective. By the above, it suffices to show that f is injective if and only if for any subset $A \subset X$, we have

$$f(X \setminus A) \subset Y \setminus f(A).$$

Suppose that the inclusion of sets is true, and take $A = \{a\}$. Then, for any $x \in X, x \neq a$, we have that $f(b) \in Y \setminus \{f(a)\}$, and f is injective. Conversly, suppose $f(X \setminus A) \not\subset Y \setminus f(A)$. Then, there exist $x \notin A$ such that $f(x) \in f(A)$, and f is not injective.

2. Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = Id_{\mathbb{R}}$ and $g \circ f \neq Id_{\mathbb{R}}$.

Solution: Let $f(x) = \arctan(x)$ which is defined to be the unique angle y with $-\pi/2 < y < \pi/2$ and $\tan(y) = x$. Now, define

$$g(y) = \begin{cases} \tan(y) & -\pi/2 < y < \pi/2 \\ 10 & \text{otherwise} \end{cases}$$

Then, for all $x \in \mathbb{R}$, $g \circ f(x) = x$, and for all $-\pi/2 < y < \pi/2$, $f \circ g(y) = y$, but $f \circ g(100) = \tan(10) \neq 100$.

If we lift the restriction that both domains and images are $\mathbb{R}$, we can also give an example with $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = x^2$ and

$g(y) = \sqrt{y}$, where $\sqrt{y}$ is defined to be the unique real number $x \geq 0$ such that $x^2 = y$. Then, $f \circ g(y) = y$ for all $y \in \mathbb{R}^+$, but $g \circ f(-2) = g(4) = 2 \neq -2$.

3. (a) Does the the subset $\mathcal{S} \subset \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^3 = y^2\}$ define a function?
(b) Does the the subset $\mathcal{S} \subset \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^3\}$ define a function ?

Solutions:

(a) Now, because the tuples $(4,-8)$ and $(4,8)$ both belong to $\mathcal{S}$, so the function would have 2 values at the point 4.

(b) Yes, it corresponds to the function $y = x^{2/3} = \sqrt[3]{x^2}$, and this is well defined since $x = y^3$ is a bijection on $\mathbb{R}$.

4. Suppose R is an equivalence relation on the set A; for each $x \in A$ define $[x] = \{y \in A : \; xRy\}$. Show that either $[x] = [z]$ or $[x] \cap [z] = \emptyset$. The sets $[x]$ are called **equivalence classes** and the collection of equivalence classes of an equivalence relation R on a set A is called the **quotient set**, denoted by A/R.

Solutions: Let $x, z \in A$, and suppose that $[x] \cap [z] \neq \emptyset$. We have to show that $[x] = [z]$. Let $y \in [x] \cap [z]$. Then, xRy and zRy, and by symmetry and transitivity, xRy and yRz and then xRz. Then, for any $x' \in [x]$, we have $x'Rx$ and xRy which implies that $x'Ry$ and $x' \in [y]$. This shows that $[x] \subset [y]$, and similarly we show that $[y] \subset [x]$, so $[x] = [y]$.

5. Let $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, and define the relation $R \subseteq S \times S$ by $(a,b)R(a',b')$ if and only if $ab' - a'b = 0$. Show that R is an equivalence relation and describe the equivalence classes.

Solutions: We have to show that the relation is reflexive, symmetric and transitive.

- Reflexive: $(a,b)R(a,b) \iff ab - ab = 0$ which is true.
- Symmetric: $(a,b)R(a',b') \iff ab' - a'b = 0 \iff a'b - ab' = 0 \iff (a',b')R(a,b)$, and the relation is symmetric.
- Transitive:

$$(a,b)R(a',b') \text{ and } (a',b')R(a'',b'') \iff ab' - a'b = 0 \text{ and } a'b'' - a''b' = 0.$$

Multiplying the first equation by b'' and the second by b and adding, we get

$$\begin{aligned} 0 &= b''(ab' - a'b) + b(a'b'' - a''b') = b''ab' - b''a'b + ba'b'' - ba''b' \\ &= b''ab' - ba''b' = b'(ab'' - a''b), \end{aligned}$$

and $ab'' - a''b = 0 \iff (a,b)R(a'',b'')$.

6. We recall that a set A is countable if either A is finite, or $\text{card}(A) = \text{card}(\mathbb{N})$.

(a) Show that if A, B are countable, so are $A \times B$ and $A \cup B$.

(b) Show that if $A_1, \dots, A_k$ are countable, so are $A_1 \times \cdots \times A_k$ and $A_1 \cup \cdots \cup A_k$.

(c) Use this to complete the proof that $\mathbb{Q}$ is countable.

Solutions:

(a) If A, B are finite, it is obvious that $A \cup B$ and $A \times B$ are finite.

If A is infinite and B is countable (finite or infinite), we first show that $\text{card}(A \times B) = \text{card}(\mathbb{N})$. By hypothesis, there exists a bijection $f_1 : A \to \mathbb{N}$ and an injection $f_2 : B \to \mathbb{N}$ (which can be a bijection of course if B is infinite), and then an injection between $A \times B$ and $\mathbb{N} \times \mathbb{N}$ given by $f((a,b)) = (f_1(a), f_2(b))$, and $\text{card}(A \times B) \leq \text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. For the reverse equality, we have that f_1^{-1} gives an injection of $\mathbb{N}$ in $A \times B$ by $g(n) = (f^{-1}(n), b)$ where b is any element of B, and we get that $\text{card}(\mathbb{N}) \leq \text{card}(A \times B)$. Putting both together, we get that $\text{card}(\mathbb{N}) = \text{card}(A \times B)$.

We now show that $A \cup B$ is countable. By hypothesis on A and B, there is a bijection $f : A \to \mathbb{N}$ and an injection $g : B \to \mathbb{N}$. Then $g(n) = f^{-1}(n)$ is an injection of $\mathbb{N}$ in $A \cup B$, and $\text{card}(\mathbb{N}) \leq \text{card}(A \cup B)$. Now, there is an injection $h : A \cup B \to \mathbb{N} \times \mathbb{N}$ given by

$$\begin{aligned} h(x) &= (f(x), 0) \quad \text{if } x \in A \\ &= (0, g(x)) \quad \text{if } x \in B \setminus A \end{aligned}$$

and $\text{card}(A \times B) \leq \text{card}(\mathbb{N} \times \mathbb{N}) = \text{card}(\mathbb{N})$. Putting the 2 results together, we have that $\text{card}(\mathbb{N}) = \text{card}(A \cup B)$.

(b) Using (a) and induction on k. If $k = 2$, the result is true. Suppose that $k \geq 3$, and that the result is true for $k - 1$. We have

$$\text{card}\left(A_1 \times A_2 \times \cdots \times A_k\right) = \text{card}\left(A_1 \times (A_2 \times \cdots \times A_k)\right) = \text{card}(\mathbb{N})$$

by induction, since we have that $A_2 \times \cdots \times A_k$ is countable by induction hypothesis. The proof for $A_1 \cup \cdots \cup A_k$ is the same.

7. Let $A = B = \mathbb{Z}$. We apply the Schroder-Bernstein Theorem to the maps $f : A \to B$ and $g : B \to A$ given by $f(a) = 2a$ and $g(b) = 3b$ to build a bijection $h : A \to B$.

(a) Show that f and g are injections (which is necessary to apply the theorem).

(b) Describe the sets A_k and B_k, $k \geq 1$. For example,

$$f(A) = \{2a : a \in \mathbb{Z}\}.$$

(c) Find $h(n)$ for $1 \leq n \leq 30$ and $n = 18, 36, 54, 108, 216$ and 324.

(d) Write a general formula for $h(n)$ for any integer n.

Solutions:

(a) If $2a = 2b$, then $2(a-b) = 0$ which implies that $a - b = 0 \iff a = b$, and f is injective. Same for g.

(b) We have

$$\begin{aligned} B_1 &= \mathbb{Z} \setminus f(A) = \{2a+1 : a \in \mathbb{Z}\} \\ A_1 &= g(B_1) = \{3(2a+1) = 6a+3 : a \in \mathbb{Z}\} \\ B_2 &= f(A_1) = \{2(6a+3) = 12a+6 : a \in \mathbb{Z}\} \\ A_2 &= g(B_2) = \{3(12a+6) = 36a+18 : a \in \mathbb{Z}\} \end{aligned}$$

and in general for any $k \geq 1$,

$$\begin{aligned} B_k &= \left\{6^{k-1}(2a+1) : a \in \mathbb{Z}\right\} \\ A_k &= \left\{3 \cdot 6^{k-1} \cdot (2a+1) : a \in \mathbb{Z}\right\} \end{aligned}$$

(c) We have that $h(x) = g^{-1}(x)$ for $x \in A_k$ for some $k \geq 1$, and $x = f(x)$ if $x \notin A_k$ for any k.

For all $1 \leq n \leq 30$, we have that $3, 9, 15, 21, 27 \in A_1$, and then $f(3) = g^{-1}(3) = 1$, $f(9) = g^{-1}(9) = 3$, $f(15) = g^{-1}(15) = 5$, $f(21) = g^{-1}(21) = 7$, $f(27) = g^{-1}(27) = 9$. Also, $18 = 3 \cdot 6 \cdot 1 \in A_2$, and then $f(18) = g^{-1}(18) = 6$. All other $1 \leq n \leq 30$ are not in A_k for any $k \geq 1$, and then $h(n) = f(n) = 2n$.

For the other values, we have $36 = 3 \cdot 6 \cdot 2$ and $216 = 3 \cdot 36 \cdot 2 \notin A_k$ for any k, so $h(36) = 72$ and $h(216) = 432$. Also, $54 = 3 \cdot 6 \cdot 3 \in A_2$, $108 = 3 \cdot 36 \cdot 1 \in A_3$, and $324 = 3 \cdot 36 \cdot 3 \in A_3$, so $h(54) = 18$, $h(108) = 36$ and $h(324) = 108$.