Important Notes : Chapter 3

- $(F|P,i,N)$: this is Compound Amount Factor
  read as F, given P, i, N

- It is useful to think of the notation $(F|P,i,N)$ as:
  "what is equivalent ($) F for ($) P given i and N?"

- So, for $(F|P,i,N)$ it is always (unknown | knowns)
  - What you are trying to do is to use elements of known set (i.e. P, i and N) to discover the unknown variable (here the F)
  - The other handy way to think it as, with $(F|P,i,N)$ you are trying to convert ($)P into ($) F (hence the notation $F \leftarrow P$) using the given interest rate i and the number of compounding period N.

- So, $(F|P,i,N)$: Compound Amount Factor; $F \leftarrow P$; Convert P to F, given i, N

  $(P|F,i,N)$: Present Worth Factor; $F \rightarrow P$; convert F to P, given i, N

  $(A|F,i,N)$: Sinking Fund Factor; $F \rightarrow A$; convert F to P, given i, N

  $(F|A,i,N)$: Uniform Series Compound Amount Factor; $F \leftarrow A$; convert A to F, given i, N

  $(A|P,i,N)$: Capital Recovery Factor; $P \rightarrow A$; convert P to A, given i, N

  $(P|A,i,N)$: Series Present Worth Factor; $P \leftarrow A$; Convert A to P, given i, N

  $(A|G,i,N)$: Arithmatic Gradient to Annuity Conversion Factor; $G \rightarrow A$; Convert G to A, given, i, N

  $(P|A,g,i,N)$: Geometric Gradient to Present Worth Conversion Factor; $P \leftarrow (A,g)$; Convert A to P, given g, i, N

Compound Amount Factor $(F|P,i,N)$

Here note that, $\boxed{F = P\cdot(1+i)^N}$ ——— (Eq 1)

ie. Future Worth = Present Worth × Compound Amount Factor $(F|P,i,N)$

How?

Notice that,

Future worth now (at time 0) is $F_0 = P$

" " at time 1 is $F_1 = \{P(1+i)\}$

" " " 2 " $F_2 = \{\cdot\}(1+i) = P(1+i)^2$

⋮

" " " N " $F_N = P(1+i)^{N-1}\cdot(1+i) = P(1+i)^N$

So, sum of all future worth, $F = F_0 + F_1 + F_2 + \cdots + F_N$

$$= \sum_{K=1}^{N} F_K$$

$$\boxed{F = P(1+i)^N}$$

where, $(1+i)^N$ = Compound Amount Factor

$$\boxed{(1+i)^N \equiv (F|P,i,N)}$$

Present Worth Factor $(P|F,i,N)$

Notice from (Eq 1): $F = P(1+i)^N$

$$\Rightarrow P = F\cdot\frac{1}{(1+i)^N} = F\cdot(P|F,i,N)$$

Therefore, Present Worth Factor is $(P|F,i,N) \equiv \frac{1}{(1+i)^N}$
which is the inverse of the compound Amount Factor

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where $\left\{ \frac{1}{(1+i)^N} \right\} = (P|F, i, N)$

& is called the present worth factor.

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* A = Annuities → disbursements of equal size A.

eg. Mortgage, Lease Payments, maintenance contracts.

A sinking fund factor $(A|F, i, N)$ converts a future amount F into an equal sized amount A for N periods at i.

If a series of payments A follows a pattern of a standard annuity of N payments in length, then the future value of payment in the jth period is given by,

$$F = A(1+i)^{N-j}$$

Then future value of all payments

$$F = A(1+i)^{N-1} + A(1+i)^{N-2} + \cdots + A(1+i)^{1} + A$$

$$\Rightarrow F = A\left[(1+i)^{N-1} + (1+i)^{N-2} + \cdots + (1+i) + 1\right]$$

→ (A)

(1) × (1+i)

$$\Rightarrow F(1+i) = A\left[\qquad\qquad\right](1+i)$$

$$= A\left[(1+i)^N + (1+i)^{N-1} + \cdots + (1+i)^2 + (1+i)\right] \quad \text{—— (2)}$$

Doing (2) − (1) gives:

$$\begin{aligned} F(1+i) &= A\left[(1+i)^N + \cancel{(1+i)^{N-1}} + \cancel{(1+i)^{N-2}} + \cdots + \cancel{(1+i)^2} + \cancel{(1+i)}\right] \\ -\ F &= A\left[\qquad\quad \cancel{(1+i)^{N-1}} + \cancel{(1+i)^{N-2}} + \cdots + \cancel{(1+i)^2} + \cancel{(1+i)} + 1\right] \end{aligned}$$

$$\Rightarrow \cancel{F} + Fi - \cancel{F} = A\left[(1+i)^N - 1\right]$$

$$\Rightarrow Fi = A\left[(1+i)^N - 1\right]$$

$$\Rightarrow A = F\left[\frac{i}{(1+i)^N - 1}\right]$$

$$= F \cdot \underbrace{(A|F, i, N)}_{\text{sinking fund factor}}$$

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* Then uniform series compound amount factor, $(F|A, i, N)$ converts an annuity A into a future amount F

So, if $A = F \cdot (A|F, i, N)$

$$\Rightarrow F = A \cdot \left[\frac{1}{(A|F, i, N)}\right] = A \cdot (F|A, i, N)$$

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* Capital Recovery Factor, $(A|P, i, N)$

– (gives us) how much money, of equal size A, must be saved over N future periods to "recover" a capital investment of P today.

Intuitively,

$(A|P, i, N)$ → converts a pesent amount P into N equal future amounts of A at i%.

Notice $\underbrace{(F|P, i, N)}_{\text{compound amount factor}}$ → coverts P into F

and $\underbrace{(A|F, i, N)}_{\text{sinking fund factor}}$ → converts F into equal sized A.

$$\therefore (A|P, i, N) = \{(F|P, i, N)\} \cdot [(A|F, i, N)]$$

$$= \{ (1+i)^N \} \cdot \left[\frac{i}{(1+i)^N - 1} \right]$$

$$(A|P, i, N) = \frac{i(1+i)^N}{(1+i)^N - 1}$$

How do we get this:

Note that if a series of equal-sized payments A follows the pattern of a standard annuity of N payments in length, then the present value of the payments in the jth period is.

$$P = A \cdot \frac{1}{(1+i)^j}$$

The PV of the total of all the annuity is then.

$$P = A\left(\frac{1}{(1+i)}\right) + A\left(\frac{1}{(1+i)^2}\right) + \cdots$$
$$+ A\left(\frac{1}{(1+i)^{N-1}}\right) + A\left(\frac{1}{(1+i)^N}\right) \qquad \text{——— (3)}$$

Doing (3) $\times$ $(1+i)$ $\Rightarrow$

$$P(1+i) = A \cdot 1 + A\left(\frac{1}{(1+i)}\right) + \cdots$$
$$+ A\left(\frac{1}{(1+i)^{N-2}}\right) + A\left(\frac{1}{(1+i)^{N-1}}\right) \qquad \text{——— (4)}$$

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Then (4) − (3) ⇒

$$P(1+i) = A\left[1 + \cancel{\frac{1}{(1+i)}} + \cancel{\frac{1}{(1+i)^2}} + \cdots + \cancel{\frac{1}{(1+i)^{N-2}}} + \cancel{\frac{1}{(1+i)^{N-1}}}\right]$$

$$-\quad P = A\left[\cancel{\frac{1}{(1+i)}} + \cancel{\frac{1}{(1+i)^2}} + \cdots + \cancel{\frac{1}{(1+i)^{N-2}}} + \cancel{\frac{1}{(1+i)^{N-1}}} + \frac{1}{(1+i)^N}\right]$$

$$\cancel{P} + Pi - \cancel{P} = A\left[1 - \frac{1}{(1+i)^N}\right]$$

$$\Rightarrow \quad P \cdot i = A\left[\frac{(1+i)^N - 1}{(1+i)^N}\right]$$

$$\Rightarrow \quad A = P \cdot \left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$

$$= P \cdot \underbrace{(A \mid P, i, N)}_{\text{Capital recovery factor}}$$

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* Series Present Worth Factor,

$(P|A,i,N)$ → converts an annuity A into present value P.

This is the inverse of the capital recovery factor $(A|P,i,N)$

$$\therefore (P|A,i,N) = \frac{1}{(A|P,i,N)}$$

$$= \frac{(1+i)^N - 1}{i(1+i)^N}$$

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* Arithmetic Gradient Series

$S_1 = 0$, $S_2 = G$, $S_3 = 2G$, ..., $S_N = (N-1)G$

or, $S_1' = A$, $S_2' = A+G$, $S_3' = A+2G$, ... , $S_N' = A+(N-1)G$

Idea is $$S_3' - S_2' = (A+2G) - (A+G)$$
$$= G$$

or, $$S_{i+1}' - S_i' = G$$

Where do we use it?

— Model a pattern of increasing operating cost for an aging plant.

Arithmetic Gradient to Annuity conversion factor

$(A|G, i, N)$ → converts G to A.

and is given by,

$$(A|G, i, N) = \left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]$$

How do we get it?

Pic 3.4

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Note that cash flow amount consists of G at the end of period 2,

$2G$ " " " " " 3

⋮

$(N-1)G$ " " " " " N

- Then future value of G over $(N-2)$ period is $G(1+i)^{N-2}$, why $(N-2)$ period b'cos total period is N & G occurs at the end of period 2.

$$\therefore F = \left[G(1+i)^{N-2} + 2G(1+i)^{N-3} + \cdots + (N-2)G(1+i) + (N-1)G\right] \quad \text{--- (5)}$$

Doing $(5) \times (1+i) \Rightarrow$

$$F(1+i) = [\cdot]\,(1+i)$$

$$= G(1+i)^{N-1} + 2G(1+i)^{N-2} + \cdots + (N-2)G(1+i)^2 + (N-1)G(1+i) \quad \text{--- (6)}$$

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Then (6) − (5) ⇒

$$F(1+i) = G(1+i)^{N-1} + 2G(1+i)^{N-2} + \cdots + (N-2)G(1+i)^2 + (N-1)G(1+i)$$

$$-F = -G(1+i)^{N-2} - \cdots - (N-3)G(1+i)^2 - (N-2)G(1+i) - (N-1)G$$

$$F.i = G(1+i)^{N-1} + G(1+i)^{N-2} + G(1+i)^2 + G(1+i) \underbrace{-(N-1)G}_{-NG+G}$$

$$= G\left[(1+i)^{N-1} + (1+i)^{N-2} + \cdots + (1+i) + 1\right] - NG$$

Note that [·] is series compound amount factor,

$$(F|A,i,N) = \left[1 + (1+i) + \cdots + (1+i)^{N-2} + (1+i)^{N-1}\right]$$

$$\therefore F.i = G\cdot(F|A,i,N) - NG \quad \text{— (7)}$$

(7) × (A|F,i,N) ⇒ ← sinking fund factor

$$i\cdot\{F(A|F,i,N)\} = G\cdot\cancel{(F|A,i,N)\cdot(A|F,i,N)} - NG(A|F,i,N)$$

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$$\Rightarrow i \cdot A = G - NG(A|F,i,N)$$

$$\Rightarrow i \cdot A = G\left[1 - N(A|F,i,N)\right]$$

$$\Rightarrow A = G\left[\frac{1}{i} - \frac{N \cdot (A|F,i,N)}{i}\right]$$

$$\cancel{\Rightarrow A = G\left[\frac{1}{i} - \frac{N(A|F,i,N)}{i}\right]}$$

$$\Rightarrow A = G\left[\frac{1}{i} - \frac{N \cdot \left\{\frac{i}{(1+i)^N - 1}\right\}}{i}\right]$$

$$\boxed{\Rightarrow A = G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]}$$

→ Arithmetic Gradient to Annuity conversion factor.

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* Conversion Factor for Geometric Gradient Series

Here is the series

$S_1 = 1, \quad S_2 = (1+g), \quad S_3 = (1+g)^2 \ldots, \quad S_N = (1+g)^{N-1}$

or, $S_1' = A, \quad S_2' = A(1+g), \quad S_3' = A(1+g)^2 \ldots, \quad S_N = A(1+g)^{N-1}$

Idea is $\quad S_3' / S_2' = \dfrac{A(1+g)^2}{A(1+g)} = (1+g)$

Where do we use it?

- to model the effects of inflation or deflation
- to model " " " productivity improvement or degradation
- to " " " " market share expansion/shrinkage

Geometric Gradient to Present Worth conversion factor

Pic 3.8

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Present worth of a geometric series:

A at the end of period 1; so, present worth $\frac{A}{(1+i)}$

$A(1+g)$ " " " 2; " " " $\frac{A(1+g)}{(1+i)^2}$

$\vdots$

$A(1+g)^{N-1}$ " " . . . . N; " " $\frac{A(1+g)^{N-1}}{(1+i)^N}$

So, total present worth:

$$P = \left[\frac{A}{(1+i)} + \frac{A(1+g)}{(1+i)^2} + \cdots + \frac{A(1+g)^{N-1}}{(1+i)^N}\right] \qquad (8)$$

$$(8) \times \frac{(1+g)}{(1+g)} \Rightarrow$$

$$P = \frac{(1+g)}{(1+g)}\left[\cdot\right]$$

$$= \frac{A}{(1+g)}\left[\frac{(1+g)}{(1+i)} + \frac{(1+g)^2}{(1+i)^2} + \cdots + \frac{(1+g)^N}{(1+i)^N}\right]$$

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Now define, growth-adjusted interest rate i^0 as:

$$i^0 = \frac{(1+i)}{(1+g)} - 1 \qquad \text{—————} (9)$$

$$= \frac{(1+i)-(1+g)}{(1+g)} = \frac{i-g}{(1+g)} = \frac{1}{(1+g)}\{i-g\}$$

Note from (9) ⇒

$$i^0 = \frac{(1+i)}{(1+g)} - 1$$

$$\Rightarrow \quad i^0 + 1 = \frac{(1+i)}{(1+g)}$$

$$\Rightarrow \quad \frac{1}{(1+i^0)} = \frac{(1+g)}{(1+i)} \qquad \text{—————} (10)$$

Use (10) into (8) to get:

$$P = \frac{A}{(1+g)}\underbrace{\left[\left(\frac{1}{1+i^0}\right) + \frac{1}{(1+i^0)^2} + \cdots + \frac{1}{(1+i^0)^N}\right]}$$

This is simply the present worth of an annuity of $\left(\frac{A}{1+g}\right)$ amount over N periods with i^0

$$\therefore \cancel{P = \frac{A}{(1+g)} (P|A}$$

$$\therefore P = A \left\{ \frac{P|A, i^\circ, N)}{(1+g)} \right\}$$

so that, $$(P|A, g, i, N) = \frac{(P|A, i^\circ, N)}{(1+g)}$$

$$= \underbrace{\left[\frac{(1+i^\circ)^N - 1}{i^\circ (1+i^\circ)^N} \right] \frac{1}{(1+g)}}$$

Geometric Gradient to Present worth conversion factor.