COMP 232 Mathematics for Computer Science
Winter 2016
Midterm Exam

Name: ______________________ Total Points:

ID: ______________________ ________ / 64

Instructions. *This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet.* *Use pen, not pencil.* *Do not detach any pages from this exam!*

($2^{\text{pts}}_{\text{ea.}}$) **1.** For each of the following propositional sentences, state whether or not it is a tautology. [ ] 10 pts

(a) $(p \wedge q) \vee \big(\neg p \vee (p \wedge \neg q)\big)$

☐ Not tautology ☑ Tautology ☐ Don't know!

(b) $(p \wedge q) \wedge (\neg p \vee q)$

☐ Tautology ☑ Not a tautology ☐ Don't know!

(c) $\big((p \to q) \wedge (q \to p)\big) \leftrightarrow (p \leftrightarrow q)$

☐ Not a tautology ☑ Tautology ☐ Don't know!

(d) $\big((p \to q) \wedge (\neg r \to \neg q)\big) \to \big(p \to (q \wedge r)\big)$

☑ Tautology ☐ Not a tautology ☐ Don't know!

(e) $\big(p \to (q \to r)\big) \leftrightarrow \big((p \to q) \to r\big)$

☑ Not a tautology ☐ Tautology ☐ Don't know!

[ ] 10 pts

(6pts) **2.** Here you are to prove a propositional equivalence using the laws in the handout. Note that you have to mention explicitly any use of commutativity, associativity, or double negation.

6 pts

In the table below, construct a proof of the equivalence

$$\neg(p \to q) \to p \equiv True$$

Step			Law applied
$\neg(p \to q) \to p$	$\equiv$	$\neg\neg(p \to q) \vee p$	law for conditional
	$\equiv$	$(p \to q) \vee p$	double negation
	$\equiv$	$(\neg p \vee q) \vee p$	law for conditional
	$\equiv$	$p \vee (\neg p \vee q)$	commutativity
	$\equiv$	$(p \vee \neg p) \vee q$	associativity
	$\equiv$	$T \vee q$	law of excluded middle
	$\equiv$	$True$	domination

6 pts

(2 pts ea.) **3.** Assume that the universe of discourse for x is the set of all people, and for y the set of all movies. Let $S(x,y)$ mean "x saw y," $L(x,y)$ mean "x liked y," $A(y)$ mean "y won an award," $C(y)$ mean "y is a comedy," and $D(x,y)$ mean "x directed y."

8 pts

(a) No comedy won an award.

Answer:

$\forall x \Big(C(x) \rightarrow \neg A(x) \Big)$ or $\neg \exists x \Big(C(x) \wedge A(x) \Big)$.

(b) No one liked every movie he/she has seen.

Answer:

$\forall x \exists y \Big(S(x,y) \wedge \neg L(x,y) \Big)$.

(c) Mary has never seen a movie that won an award.

Answer:

$\forall y \Big(S(Mary, y) \rightarrow \neg A(y) \Big)$ or $\neg \exists y \Big(S(Mary, y) \wedge A(y) \Big)$.

(d) Someone has seen at least one movie by every director.

Answer:

$\exists x \forall x' \exists y \Big(D(x', y) \wedge S(x,y) \Big)$

8 pts

(2 pts ea.) **4.** Let $P(x, y)$ be a predicate and let the Universe of Discourse for the variables x and y be $\{1, 2, 3\}$. Suppose $P(1, 3), P(2, 1), P(2, 2), P(2, 3), P(3, 1), P(3, 2)$ are true, and $P(x, y)$ is false otherwise. Determine whether the following statements are true or false.

12 pts

Determine the truth value of each of the following statements

(a) $\forall x \exists y \left(P(x, y)\right)$

- [x] True
- [ ] False
- [ ] Don't know!

(b) $\exists x \forall y \left(P(x, y)\right)$

- [ ] False
- [x] True
- [ ] Don't know!

(c) $\neg\left(\exists x \exists y \left(P(x, y) \wedge \neg P(y, x)\right)\right)$

- [x] False
- [ ] True
- [ ] Don't know!

(d) $\forall y \exists x \left(P(x, y) \rightarrow P(y, x)\right)$

- [x] True
- [ ] False
- [ ] Don't know!

(e) $\forall x \forall y \left((x = y) \rightarrow \left(P(x, y) \vee P(y, x)\right)\right)$

- [ ] True
- [x] False
- [ ] Don't know!

(f) $\forall y \exists x \left((x \leq y) \wedge P(x, y)\right)$

- [x] False
- [ ] True
- [ ] Don't know!

12 pts

(2 pts ea.) **5.** Express the negation of each of the statements (a) to (e) below, so that all negation symbols immediately precede predicates.

10 pts

(a) $\forall x\, \exists y\, \forall z\, T(x, y, z)$

Negated:

$\exists x \forall y \exists z\ \neg T(x, y, z)$

(b) $\forall x \exists y P(x, y) \lor \forall x \exists y Q(x, y)$

Negated:

$\exists x \forall y\ \neg P(x, y)\ \land\ \exists x \forall y\ \neg Q(x, y)$

(c) $\exists x \exists y \Big(Q(x, y) \leftrightarrow Q(y, x) \Big)$

Negated:

$\forall x \forall y \Big[\Big(Q(x, y) \land \neg Q(y, x) \Big) \lor \Big(Q(y, x) \land \neg Q(x, y) \Big) \Big]$

(d) $\forall y \exists x \exists z\ \Big(T(x, y, z) \lor Q(x, y) \Big)$

Negated:

$\exists y \forall x \forall z\ \Big(\neg T(x, y, z) \land \neg Q(x, y) \Big)$

(e) $\neg \Big(\forall x \Big(\exists y \forall z T(x, y, z) \land \exists z \forall y T(x, y, z) \Big) \Big)$

Negated:

$\forall x \Big(\exists y \forall z T(x, y, z) \land \exists z \forall y T(x, y, z) \Big) \Big)$

10 pts

(6 pts) **6.** Suppose the set of premises is

$$\{p \to q,\ \neg p \to s,\ r \to s\}.$$

Then $\neg q \to s$ is a valid conclusion from the premises. This can be proved as follows:

No.	**Step**	**Reason**
1	$p \to q$	Premise
2	$\neg q \to \neg p$	Contrapositive of (1)
3	$\neg p \to r$	Premise
4	$\neg q \to r$	Hypothetical syllogism using (2) and (3)
5	$r \to s$	Premise
6	$\neg q \to s$	Hypothetical syllogism using (4) and (5)

Similarly to the above, prove that from premises

$$\{(p \wedge q) \vee r,\ r \to s\}$$

we can get $p \vee s$ as valid conclusion. Use the rules of inference (and possibly equivalences) from the crib-sheet.

No.	**Step**	**Reason**
1	$(p \wedge q) \vee r$	Premise
2	$(p \vee r) \wedge (q \vee r)$	Distributivity on (1)
3	$p \vee r$	Simplification from (2)
4	$r \to s$	Premise
5	$\neg r \vee s$	Law for conditional on (4)
6	$r \vee p$	Commutativity on (3)
7	$s \vee p$	Resolution on (5) and (6)
8	$p \vee s$	Commutativity on (7)

6 pts

6 pts

($3^{\text{pts}}_{\text{ea.}}$) **7.** (a) Consider the assertion

"*Let m and n be integers. If $m+n$ is even, then $m-n$ is even.* "

In the box below, give a direct proof of the assertion.

12 pts

If $m+n$ is even, then $m+n = 2k$, for some $k \in \mathbb{Z}$.
Thus $m = 2k - n$ and $m - n = 2k - n - n = 2k - 2n = 2(k-n)$
$\Rightarrow m - n$ is even.

(b) Consider the assertion "*Let $n \in \mathbb{Z}$. If $n^5 + 7$ is even, then n is odd.* "
In the box below, give a indirect proof of the assertion.

Suppose to the contrary that n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$.
We get $n^5 + 7 = (2k)^5 + 7 = 32k^5 + 7 = 32k^5 + 6 + 1 = 2(16k^5 + 3) + 1$
$\Rightarrow n^5 + 7$ is odd.

(c) Consider the assertion

"*Suppose that n is rational $n \neq 0$, and that m is irrational, $m \neq 0$. Then $n \cdot m$ is irrational.* "

In the box below, prove the assertion by contradiction.

The counter-assumption is n is rational, m is irrational, and $n \cdot m$ is rational.
n is rational $\Rightarrow n = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$.
$n \cdot m$ is rational $\Rightarrow n = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$, $d \neq 0$.
$\Rightarrow m = \frac{n \cdot m}{n} = \frac{\frac{c}{d}}{\frac{a}{b}} = \frac{c \cdot b}{d \cdot a}$
We have $n \neq 0 \Rightarrow a \neq 0$ and we assumed $d \neq 0$.
$\Rightarrow m = \frac{c \cdot b}{d \cdot a}$ is rational; a contradiction.

(d) Consider the assertion

"*For all integers n, if 3 does not divide n, then n^2 leaves remainder 1 when it is divided by 3.* "

In the box below, give a proof by cases of the assertion.

- Case 1: $n \bmod 3 = 1$ (meaning remainder is 1)

$\Rightarrow n = 3k + 1$ for some $k \in \mathbb{Z}$
$\Rightarrow n^2 = 9k^2 + 6k + 1 = 3(3k^3 + 2k) + 1$
$\Rightarrow n^2 \bmod 3 = 1$

- Case 2: $n \bmod 3 = 2$ (meaning remainder is 2)

$\Rightarrow n = 3k + 2$ for some $k \in \mathbb{Z}$
$\Rightarrow n^2 = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 = 3(3k^3 + 2k) + 1$
$\Rightarrow n^2 \bmod 3 = 1$

— End of Exam —

12 pts