1. If $\{u, v, w\}$ is a set of vectors in a vector space V, and a, b, and c are scalars, which of the following statements are true?

I. If $a = b = c = 0$ implies $au + bv + cw = 0$, then $\{u, v, w\}$ is linearly independent.

II. If $au + bv + cw = 0$ implies $a = b = c = 0$, then $\{u, v, w\}$ is linearly independent.

III. If none of the vectors u, v or w is a multiple of any other vector in $\{u, v, w\}$, then $\{u, v, w\}$ is linearly independent.

IV. If $au + bv + cw = 0$ can occur only when $a = b = c = 0$, then $\{u, v, w\}$ is linearly independent.

A. I & II
B. I & III
C. II & III
D. I & IV
E. II & IV
F. III & IV

I. is false: the statement "$a=b=c=0 \Rightarrow au+bv+cw=0$" holds for any vectors u, v, w (e.g. $u=v=w=0$).

II is true: it is the definition of linear independence

III. is false: $\{(1,0),(0,1),(1,1)\}$ satisfies the given condition, but is l. dependent.

IV is true: this is simply a restatement of the definition: "A implies B" means A can only occur when B does too.

2. For what value of a is the set of vectors $\{(1,1,1), (1,0,2), (1,a,1)\}$ linearly dependent?

A. -1
B. 2
C. 0
D. 1
E. -1/2
F. -2

We seek values of a s.t. there are scalars b, c

s.t. $b(1,1,1) + c(1,0,2) = (1,a,1)$

i.e.
$$\begin{aligned} b + c &= 1 \quad (1)\\ b &= a \quad (2)\\ b + 2c &= 1. \quad (3)\end{aligned}$$

(1) & (3) $\Rightarrow$ $c=0$, and $b=1$, and (2) then implies $a=b=1$.

3. In a vector space V, suppose $\{u, v\}$ is linearly independent and w is such that $\{u, v, w\}$ is linearly *dependent*. Which of the following is **ALWAYS** true?

A. $\{u, w\}$ is linearly dependent

B. $\{v, w\}$ is linearly dependent

C. $u \in \text{span}\{v, w\}$

D. $v \in \text{span}\{u, w\}$

E. $w \in \text{span}\{u, v\}$

F. $\{u, u + v, w\}$ is linearly independent

A could be false. e.g. $u = (1,0)$, $v = (0,1)$ and $w = (1,1)$
$\{u,v\}$ is l.i. and $\{u,w\}$ is l.i. too

B could be false: Same example as for A

C " " " e.g. $u = (1,0)$, $v = (0,1)$ & $w = (0,0)$

D " " " " " "

E is true! It's a theorem we proved in class:
if $\{u,v\}$ is l.i. then $\{u,v,w\}$ is l.i. $\Leftrightarrow$
$w \notin \text{span}\{u,v\}$.

F. could be false. See the example for A

4. Let $u = (1, 0, -1)$, and $U = \{w \in \mathbf{R}^3 \mid \operatorname{proj}_u w = 0\}$.

1 a) Show that $U = \{(x, y, z) \in \mathbf{R}^3 \mid x - z = 0\}$; (1)

$2\frac{1}{2}$ b) Give a complete geometric description of U. ($1\frac{1}{2}$)
Is U a subspace of $\mathbf{R}^3$? ($\frac{1}{2}+\frac{1}{2}$)

1 c) Show that $\{(1, 0, 1), (0, 1, 0\}$ spans U. (1)

$1\frac{1}{2}$ d) What is the dimension of U? ($1\frac{1}{2}$)

a) Note that (for $w = (x,y,z)$) $\operatorname{proj}_u w = 0$ is the equation $\frac{u \cdot w}{\|u\|^2} u = 0$. But as $u \neq 0$, $\frac{u \cdot w}{\|u\|^2} u = 0$

$\iff \frac{u \cdot w}{\|u\|^2} = 0 \iff u \cdot w = 0 \iff (1,0,-1)\cdot(x,y,z) = 0$

$\iff x - z = 0$. Hence $U = \{(x,y,z) \in \mathbb{R}^3 \mid x - z = 0\}$.

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b) U is the plane through the origin with normal vector $(1,0,-1)$. $3 \times (\frac{1}{2})$

U is a subspace of $\mathbb{R}^3$ ($\frac{1}{2}$) because it is a plane through the origin. (or: see (c) $U = \operatorname{span}\{(1,0,1),(0,1,0)\}$ and so by a theorem in class is a subspace of $\mathbb{R}^3$) ($\frac{1}{2}$) - reason

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c) From (a) $U = \{(z,y,z) \mid y, z \in \mathbb{R}\} = \{z(1,0,1) + y(0,1,0) \mid y,z \in \mathbb{R}\}$
$= \operatorname{Span}\{(1,0,1),(0,1,0)\}$. Hence $\{(1,0,1),(0,1,0)\}$ spans U. (1)

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d) By (c), $\{(1,0,1),(0,1,0)\}$ spans U. Moreover, $(1,0,1)$ & $(0,1,0)$ are l.i. because neither (of these 2 vectors) is a multiple of the other. Hence $\{(1,0,1),(0,1,0)\}$ is a basis for U.

So $\underline{\dim U = 2}$. ($\frac{1}{2}$) + (1) - justification

5. Let $\mathbf{F}[0,\pi] = \{f \mid f : [0,\pi] \to \mathbf{R}\}$ be the vector space of real-valued functions defined on $[0,\pi]$. Define three functions in $\mathbf{F}([0,\pi])$ by

$$f(x) = 1, \quad g(x) = \sin 2x, \quad \text{and} \quad h(x) = \cos x, \qquad \forall x \in [0,\pi],$$

and let $W = \text{span}\{f, g, h\}$.

2 a) Show that $\{f, g, h\}$ is linearly independent.

1 b) Use (a) to show that $\{f+g, f+h\}$ is linearly independent.

1½ c) Give a spanning set S for W with $S \neq \{f, g, h\}$.

1½ d) What is the dimension of W?

(knowing what to do)

a) Suppose scalars a, b, c satisfy $af + bg + ch = 0$.
Then $a + b \sin 2x + c\cos x = 0$, $\forall x \in [0,\pi]$. 1

So for $x = 0$, we obtain $a + c = 0$
$x = \frac{\pi}{4}$ " $a + b + \frac{\sqrt{2}}{2} c = 0$
$x = \frac{\pi}{2}$ " $a = 0$.
These clearly imply $a = b = c = 0$.

Hence $\{f, g, h\}$ is l.i. 1 any ~~set of eqns~~ implying (consistent with $af + bg + ch = 0$)

b) Suppose $a(f+g) + b(f+h) = 0$ ½ Then $(a+b)f + bg + bh = 0$.
But $\{f, g, h\}$ is l.i., so $a+b = 0$, $b = 0$, and $b = 0$. This implies $a = b = 0$, so $\{f+g, f+h\}$ is l.i. + ½

c) Since none of f, g, h is the zero function, $S = \{f, g, h, 0\} \neq \{f, g, h\}$,
1 — any correct answer + ½ justification.
but span$\{f, g, h, 0\}$ = span$\{f, g, h\} = W$, since $0 \in W$ already.
(or e.g. $S = \{f, g, h, f+g\}$ spans W but is not the same set as $\{f, g, h\}$.)

d) By definition of W, $\{f, g, h\}$ spans W, and by (a) is l.i. Hence $\{f, g, h\}$ is a basis of W and so $\dim W = 3$.
½ 1 - justification

6.

a) Give an example of 3 vectors u, v, w in $\mathbf{R}^2$, such that $\{u, v, w\}$ spans $\mathbf{R}^2$, and $\{u, v\}$ also spans $\mathbf{R}^2$.

Let $u = (1,0)$, $v = (0,1)$ and $w = (1,1)$.

Then $\operatorname{span}\{u, v\} = \mathbb{R}^2$ $(\forall (x,y) \in \mathbb{R}^2,$
$(x,y) = xu + yv)$

and $\operatorname{span}\{u, v, w\} = \mathbb{R}^2$ $(\forall (x,y) \in \mathbb{R}^2,$
$(x,y) = xu + yv + 0w.)$

(1) - $\operatorname{Span}\{u,v,w\} = \mathbb{R}^2$
(1) - $\operatorname{span}\{u,v\} = \mathbb{R}^2$
+(1) - correct justification

b) If U is vector space with u, v, w in U, and $\{u, v, w\}$ is linearly independent, show that $\{u, v\}$ must also be linearly independent. (**Do not give an example**: use the definition of independence to show that the statement is true in *every* vector space U.)

Suppose a, b are scalars s.t. $au + bv = 0$. (1)
Then $au + bv + 0w = 0$. (1) But $\{u, v, w\}$ is
l.i., so $a = 0$, $b = 0$ and $0 = 0$. This implies
$a = b = 0$, so $\{u, v\}$ is l.i. (1)

7. [Bonus]

a) Suppose that u, v are two non-zero vectors in $\mathbf{R}^4$ such that $u \cdot v = 0$. Prove carefully that $\{u, v\}$ is linearly independent.

Suppose a, b are scalars with $au + bv = 0$. $(*)$

Then $u \cdot (au + bv) = u \cdot 0 = 0$, i.e.

$\|u\|^2 a + b(u \cdot v) = 0$. But $u \cdot v = 0$, so

$\|u\|^2 a = 0$. Since $u \neq 0$, $\|u\|^2 \neq 0$, so $a = 0$.

Then we obtain from $(*)$ that $bv = 0$. But

Since $v \neq 0$, b must be 0. Hence $a = b = 0$, so

$\{u, v\}$ is l.i.

1 - correct and well-written

b) Give an example of two non-zero vectors u and v in $\mathbf{R}^4$ such that $\{u, v\}$ is linearly independent but $u \cdot v \neq 0$.

Let $u = \{(1, 0, 0, 0)\}$ and $v = \{(1, 1, 0, 0)\}$.

Then neither u nor v is a multiple of the other,

so $\{u, v\}$ is l.i., but $u \cdot v = 1 \neq 0$.

1/2 - correct example

1/2 - justification, well-written.