MATH 1005A
Test 3 Solutions

March 8, 2016

[Marks]

[3] 1. $\lim_{n\to\infty} \dfrac{2n^2+2n-3}{\sqrt{9n^4+3n^2+1}} =$ (a) $\dfrac{2}{3}$ (b) 0 (c) ∞ (d) $\dfrac{2}{9}$ (e) Does not exist

Answer: (a)

[3] 2. The sequence $\{r^n\}_{n=0}^{\infty}$ converges for

(a) $|r| < 1$ (b) $|r| \le 1$ (c) $|r| > 1$ (d) $|r| \ge 1$ (e) $-1 < r \le 1$

Answer: (e)

[3] 3. $\lim_{n\to\infty} \dfrac{\ln(n)}{\sqrt{n}} =$ (a) 1 (b) e (c) 0 (d) ∞ (e) Does not exist

Answer: (c)

[3] 4. $\lim_{n\to\infty} \dfrac{3n^2}{2e^n} =$ (a) $\dfrac{3}{2}$ (b) 1 (c) e (d) 0 (e) ∞

Answer: (d)

[3] 5. The sequence $\left\{2 - \dfrac{1}{n^2}\right\}_{n=1}^{\infty}$ is

(a) increasing and bounded (b) decreasing and bounded
(c) increasing and unbounded (d) decreasing and unbounded (e) None of these

Answer: (a)

[3] 6. If $2 - \dfrac{3}{n} \le a_n \le 2 + \dfrac{1}{3^n}$ for all $n \ge 1$, then $\lim_{n\to\infty} a_n =$

(a) 0 (b) 1 (c) 2 (d) $\dfrac{1}{2}$ (e) None of these

Answer: (c)

[8] 7. Find the general solution $x(t)$, $y(t)$ of the system $\left\{ \begin{array}{l} x' = x - y \\ y' = 2x + 4y \end{array} \right\}$.

Solution:

$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, $\det(\lambda I - A) = \begin{pmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 4 \end{pmatrix} = \lambda^2 - 5\lambda + 6$

$= (\lambda - 2)(\lambda - 3) = 0 \;\Rightarrow\; \lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues.

For $\lambda_1 = 2$, $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \;\Rightarrow\; a + b = 0$, and $a = 1 \;\Rightarrow\; b = -1 \;\Rightarrow$

$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Note: $\mathbf{v}_1 = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is also correct for any $c \neq 0$.

For $\lambda_2 = 3$, $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \;\Rightarrow\; 2a + b = 0$, and $a = 1 \;\Rightarrow\; b = -2 \;\Rightarrow$

$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. <u>Note:</u> $\mathbf{v}_2 = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is also correct for any $c \neq 0$.

The general solution is $\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Hence, $x(t) = c_1 e^{2t} + c_2 e^{3t}$ and $y(t) = -c_1 e^{2t} - 2c_2 e^{3t}$.

[4] 8. Given that the matrix $A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$ has the complex eigenvalue $\lambda = 1 + i$ and the corresponding eigenvector $\mathbf{v} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$, find a (real) fundamental matrix $X(t)$ for the system $\mathbf{x}' = A\mathbf{x}$.

<u>Solution:</u>

A complex solution is $\mathbf{z}(t) = e^{(1+i)t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$= e^t[\cos(t) + i\sin(t)]\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ $\Rightarrow$ two real, independent solutions are

$x(t) = e^t\left[\cos(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = e^t\begin{pmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{pmatrix}$ and

$y(t) = e^t\left[\sin(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = e^t\begin{pmatrix} \sin(t) + \cos(t) \\ \sin(t) \end{pmatrix}$, and

a fundamental matrix is $X(t) = e^t\begin{pmatrix} \cos(t) - \sin(t) & \sin(t) + \cos(t) \\ \cos(t) & \sin(t) \end{pmatrix}$.