

**Economics 325/4 B: Winter 2016 Midterm 2: (50 Minutes 100 Marks)**

1.i) [10 marks] Find the demand for labour as a function the real wage  $w$  when the production function is

$$Q = f(L) = \frac{5}{4}L^{\frac{4}{5}} + \frac{5}{3}L^{\frac{3}{5}}.$$

ii) [10 Marks] If  $f(x)$  and  $g(x)$  have elasticities  $\eta_f(x)$  and  $\eta_g(x)$  and  $h(x) = f(x)g(x)$ , prove the elasticity of the  $h(x)$  is  $\eta_f(x) + \eta_g(x)$ . iii) [10 marks] Find the elasticity of

$$f(x) = \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}} (e^{2 \ln(x)} x^2)^{\frac{1}{10}} + x^0 - x^3 \left(\frac{1}{e^{2 \ln(x)}}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}} \left(\ln\left(\exp\left(x^{\frac{1}{10}}\right)\right)\right)^2}{\sqrt{x^{\frac{2}{5}}}} \text{ for } x > 0.$$

2. Consider

$$f(x) = \frac{\ln\left(e^{x^2}\right) \left(e^{x^2}\right)^{-\frac{1}{2}}}{e^{-2 \ln(x)}}, \text{ for } x > 0.$$

i) [8 Marks] Simplify  $f(x)$ . Find the first-order condition for  $f(x)$  and solve. ii) [9 Marks] Now consider the function  $g(x) = \ln(f(x))$  where  $f(x)$  is defined above. Simplify  $g(x)$ . Find the first-order condition for  $g(x)$ , solve, and determine if this solution yields a global minimum or maximum. Use your results for  $g(x)$  to determine the global maximum or minimum for  $f(x)$ . iii) [8 Marks] Prove that the inverse function of a globally increasing convex function is globally increasing and concave.

3. i) [9 marks] Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

calculate  $AA, A^T B, AB, B^T B, (AB)^{-1}$ . From the roots of the appropriate quadratic, find the eigenvalues of  $C$ . ii) [9 marks] Now consider

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 23^{23} & 77 \\ 0 & 2 & 95 \\ 0 & 0 & 3 \end{bmatrix}.$$

Using the Laplace expansion along the first row calculate  $\det[A]$ . Calculate  $\det[B]$ . Calculate  $\det[(BA^{-1}B^T)^{-1}]$ . iii) [7 marks] Prove that  $C = (D^T D + E E^T)^{-1}$  is symmetric assuming that  $C$  exists.

4. Consider the system of equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 11 \\ x_2 + x_3 &= 5 \\ 2x_3 &= 6. \end{aligned}$$

i) [10 Marks] If these three equations are written in matrix notation as  $Ax = b$ , what are  $A$ ,  $x$ , and  $b$ ? Calculate  $x_2$  and  $x_3$  in  $Ax = b$  using Cramer's rule. ii) [10 Marks] Prove if the  $n \times 1$  vectors  $x$  and  $y$  are orthogonal then  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

**Economics 325/4 B: Winter 2016 Midterm 2: Answers**

1.i) [10 marks] Find the demand for labour as a function the real wage  $w$  when the production function is

$$Q = f(L) = \frac{5}{4}L^{\frac{4}{5}} + \frac{5}{3}L^{\frac{3}{5}}.$$

ii) [10 Marks] If  $f(x)$  and  $g(x)$  have elasticities  $\eta_f(x)$  and  $\eta_g(x)$  and  $h(x) = f(x)g(x)$ , prove the elasticity of the  $h(x)$  is  $\eta_f(x) + \eta_g(x)$ . iii) [10 marks] Find the elasticity of

$$f(x) = \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}} (e^{2 \ln(x)} x^2)^{\frac{1}{10}} + x^0 - x^3 \left(\frac{1}{e^{2 \ln(x)}}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}} \left(\ln\left(\exp\left(x^{\frac{1}{10}}\right)\right)\right)^2}{\sqrt{x^{\frac{2}{5}}}} \text{ for } x > 0.$$

**Answer:** i) [10 marks] We have

$$Q = f(L) = \frac{5}{4}L^{\frac{4}{5}} + \frac{5}{3}L^{\frac{3}{5}} \implies MP_L(L) \equiv f'(L) = L^{-\frac{1}{5}} + L^{-\frac{2}{5}}.$$

From  $MP_L(L^*) = w$  we have

$$(L^*)^{-\frac{1}{5}} + (L^*)^{-\frac{2}{5}} = w.$$

Using

$$x \equiv (L^*)^{-\frac{1}{5}} > 0 \implies x^2 = (L^*)^{-\frac{2}{5}}$$

we have

$$\begin{aligned} (L^*)^{-\frac{1}{5}} + (L^*)^{-\frac{2}{5}} &= w \\ \implies x^2 + x - w &= 0 \\ \implies (L^*)^{-\frac{1}{5}} = x &= \frac{\sqrt{1+4w} - 1}{2} \\ \implies L^* &= \left(\frac{\sqrt{1+4w} - 1}{2}\right)^{-5}. \end{aligned}$$

(We ignore the other negative root  $\frac{-\sqrt{1+4w}-1}{2}$  since  $x \equiv (L^*)^{-\frac{1}{5}} > 0$ .)

ii) [10 Marks] Given two functions  $f(x)$  and  $g(x)$  with elasticities  $\eta_f(x) = \frac{f'(x)x}{f(x)}$  and  $\eta_g(x) = \frac{g'(x)x}{g(x)}$ , and if  $h(x) = f(x)g(x)$  then the elasticity of  $h(x)$  is

$$\begin{aligned} \eta_h(x) &\equiv \frac{h'(x)x}{h(x)} = \frac{(f'(x)g(x) + f(x)g'(x))x}{f(x)g(x)} \\ &= \frac{f'(x)g(x)x}{f(x)g(x)} + \frac{f(x)g'(x)x}{f(x)g(x)} \\ &= \frac{f'(x)x}{f(x)} + \frac{g'(x)x}{g(x)} = \eta_f(x) + \eta_g(x). \end{aligned}$$

iii) [10 marks] We have

$$\begin{aligned}
 f(x) &= \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}} (e^{2\ln(x)}x^2)^{\frac{1}{10}} + x^0 - x^3 \left(\frac{1}{e^{2\ln(x)}}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}} \left(\ln\left(\exp\left(x^{\frac{1}{10}}\right)\right)\right)^2}{\sqrt{x^{\frac{2}{5}}}} \\
 &= \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}} (x^2x^2)^{\frac{1}{10}} + 1 - x^3 \left(\frac{1}{x^2}\right)^{\frac{3}{2}}\right)^{\frac{5}{3}} \left(x^{\frac{1}{10}}\right)^2}{x^{\frac{1}{5}}} \\
 &= \frac{\left(\left(\frac{3}{2}\right)^{\frac{3}{5}} x^{\frac{2}{5}} + 1 - x^3 (x^{-2})^{\frac{3}{2}}\right)^{\frac{5}{3}} x^{\frac{1}{5}}}{x^{\frac{1}{5}}} \\
 &= \left(\left(\frac{3}{2}\right)^{\frac{3}{5}} x^{\frac{2}{5}} + 1 - x^3 x^{-3}\right)^{\frac{5}{3}} \\
 &= \left(\left(\frac{3}{2}\right)^{\frac{3}{5}} x^{\frac{2}{5}}\right)^{\frac{5}{3}} = \frac{3}{2} x^{\frac{2}{3}}.
 \end{aligned}$$

Since  $f(x)$  is of the form  $Ax^b$ , the elasticity is the exponent  $\frac{2}{3}$ .

2. Consider

$$f(x) = \frac{\ln(e^{x^2}) (e^{x^2})^{-\frac{1}{2}}}{e^{-2\ln(x)}}, \text{ for } x > 0.$$

i) [8 Marks] Simplify  $f(x)$ . Find the first-order condition for  $f(x)$  and solve. ii) [9 Marks] Now consider the function  $g(x) = \ln(f(x))$  where  $f(x)$  is defined above. Simplify  $g(x)$ . Find the first-order condition for  $g(x)$ , solve, and determine if this solution yields a global minimum or maximum. Use your results for  $g(x)$  to determine the global maximum or minimum for  $f(x)$ .

iii) [8 Marks] Prove that the inverse function of a globally increasing convex function is globally increasing and concave.

**Answer:** i) [8 Marks] Simplifying we have

$$f(x) = \frac{\ln(e^{x^2}) (e^{x^2})^{-\frac{1}{2}}}{e^{-2\ln(x)}} = \frac{x^2 e^{-\frac{1}{2}x^2}}{e^{\ln(x^{-2})}} = \frac{x^2 e^{-\frac{1}{2}x^2}}{x^{-2}} = x^4 e^{-\frac{1}{2}x^2}.$$

Thus

$$f'(x) = 4x^3 e^{-\frac{1}{2}x^2} - x^5 e^{-\frac{1}{2}x^2} = x^3 e^{-\frac{1}{2}x^2} (4 - x^2)$$

and so the first-order condition is

$$(x^*)^3 e^{-\frac{1}{2}(x^*)^2} (4 - (x^*)^2) = 0.$$

We have  $(x^*)^3 > 0$  since  $x > 0$  and  $e^{-\frac{1}{2}(x^*)^2} > 0$  and so

$$4 - (x^*)^2 = 0 \implies x^* = 2.$$

(We rule out the other root  $x^* = -2$  since  $x > 0$ .)

ii) [9 Marks] For  $g(x) = \ln(f(x))$  we first simplify as

$$g(x) = \ln(f(x)) = \ln\left(x^4 e^{-\frac{1}{2}x^2}\right) = 4 \ln(x) - \frac{x^2}{2}.$$

Thus

$$g'(x) = \frac{4}{x} - x$$

and so the first-order condition  $g'(x^*) = 0$  yields

$$\frac{4}{x^*} - x^* = 0 \implies (x^*)^2 = 4 \implies x^* = 2.$$

Now

$$g''(x) = -\underbrace{\frac{4}{x^2}}_{+} < 0$$

for all  $x$  since  $x^2 > 0$ , so  $g(x)$  is globally concave, and so  $x^* = 2$  yields a global maximum for  $g(x)$ . Since  $g(x) = \ln(f(x))$  with  $\frac{d \ln(x)}{dx} = \frac{1}{x} > 0$ , or since  $f(x) = e^{g(x)}$  with  $\frac{de^x}{dx} = e^x > 0$ , it follows that  $g(x)$  and  $f(x)$  are monotonic transformations of each other, and so have the same global maximum. It follows that  $x^* = 2$  is the global maximum of  $f(x)$ .

iii) [8 Marks] If  $f(x)$  is globally increasing and concave then for all  $x$

$$f'(x) > 0 \text{ and } f''(x) < 0.$$

Since  $f'(x) > 0$  it follows that the inverse function of  $f(x)$  exists satisfying

$$f(g(x)) = x.$$

Differentiating both sides with respect to  $x$  yields

$$f'(g(x))g'(x) = 1 \implies g'(x) = \frac{1}{\underbrace{f'(g(x))}_{+}} > 0 \text{ for all } x$$

and so  $g(x)$  is globally increasing. Differentiating both sides of

$$f'(g(x))g'(x) = 1$$

with respect to  $x$  yields

$$f''(g(x))(g'(x))^2 + f'(g(x))g''(x) = 0$$

so that solving for  $g''(x)$ , (and using the fact that  $f''(x) < 0$  for all  $x$ ) shows that **for all**  $x$

$$g''(x) = -\frac{\overbrace{f''(g(x))(g'(x))^2}^{-}}{\underbrace{f'(g(x))}_{+}} > 0$$

and so  $g(x)$  is globally convex. [**Marker:** Students should indicate **for all**  $x$  for full marks.]

3. i) [**9 marks**] Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

calculate  $AA, A^T B, AB, B^T B, (AB)^{-1}$ . From the roots of the appropriate quadratic, find the eigenvalues of  $C$ . ii) [**9 marks**] Now consider

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 23^{23} & 77 \\ 0 & 2 & 95 \\ 0 & 0 & 3 \end{bmatrix}.$$

Using the Laplace expansion along the first row calculate  $\det[A]$ . Calculate  $\det[B]$ . Calculate  $\det[(BA^{-1}B^T)^{-1}]$ . iii) [**7 marks**] Prove that  $C = (D^T D + EE^T)^{-1}$  is symmetric assuming that  $C$  exists.

**Answer:** i) [**9 marks**] Here  $AA$  and  $A^T B$  are undefined. We have

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix} \\ B^T B &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \\ (AB)^{-1} &= \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}^{-1} = \frac{1}{2 \times 1 - 2 \times 4} \begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix} \\ &= \frac{1}{-6} \begin{bmatrix} 1 & -2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}. \end{aligned}$$

We have

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \implies \det[C] = 2 \times 2 - 1 \times 1 = 3 \text{ and } \text{tr}[C] = 2 + 2 = 4$$

so that

$$\begin{aligned} \lambda^2 + (-4)\lambda + 3 &= 0 \\ \implies \lambda &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} \\ \implies \lambda_1 &= 1, \lambda_2 = 3. \end{aligned}$$

ii) [**9 marks**] We have

$$\begin{aligned} \det[A] &= \det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 2 \times \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - 3 \times \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + 1 \times \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= (2 \times 1) - (3 \times 2) + (1 \times 1) = -3. \end{aligned}$$

Since  $B$  is upper triangular its determinant is the product of the diagonal elements we have

$$\det [B] = \det \begin{bmatrix} 1 & 23^{23} & 77 \\ 0 & 2 & 95 \\ 0 & 0 & 3 \end{bmatrix} = 1 \times 2 \times 3 = 6.$$

We have

$$\begin{aligned} \det [(BA^{-1}B^T)^{-1}] &= \frac{1}{\det [BA^{-1}B^T]} = \frac{1}{\det [B] \det [A^{-1}] \det [B^T]} \\ &= \frac{1}{\det [B] \frac{1}{\det [A]} \det [B]} = \frac{\det [A]}{\det [B]^2} = \frac{-3}{36} = -\frac{1}{12}. \end{aligned}$$

ii) [7 marks]  $C$  is symmetric since

$$\begin{aligned} C^T &= \left( (D^T D + EE^T)^{-1} \right)^T = \left( (D^T D + EE^T)^T \right)^{-1} \\ &= \left( (D^T D)^T + (EE^T)^T \right)^{-1} = \left( D^T (D^T)^T + (E^T)^T E^T \right)^{-1} \\ &= (D^T D + EE^T)^{-1} = C. \end{aligned}$$

4. Consider the system of equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 11 \\ x_2 + x_3 &= 5 \\ 2x_3 &= 6. \end{aligned}$$

i) [10 Marks] If these three equations are written in matrix notation as  $Ax = b$ , what are  $A$ ,  $x$ , and  $b$ ? Calculate  $x_2$  and  $x_3$  in  $Ax = b$  using Cramer's rule. ii) [10 Marks] Prove if the  $n \times 1$  vectors  $x$  and  $y$  are orthogonal then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

**Answer: i) [10 Marks]** Writing out the system of equations in matrix notation as  $Ax = b$  we have

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 5 \\ 6 \end{bmatrix}.$$

ii). Since  $A$  is upper triangular its determinant is the product of the diagonal elements or  $\det [A] = 4$ . Using Cramer's rule and going down the first column for the determinant in the numerator we have

$$x_2 = \frac{\det \begin{bmatrix} 2 & 11 & 1 \\ 0 & 5 & 1 \\ 0 & 6 & 2 \end{bmatrix}}{4} = \frac{2 \times \det \begin{bmatrix} 5 & 1 \\ 6 & 2 \end{bmatrix}}{4} = \frac{2 \times 4}{4} = 2.$$

For  $x_3$  the determinant in the numerator is also upper triangular and so we have

$$x_3 = \frac{\det \begin{bmatrix} 2 & 3 & 11 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix}}{4} = \frac{2 \times 1 \times 6}{4} = 3.$$

ii) **[10 Marks]** If the  $n \times 1$  vectors  $x$  and  $y$  are orthogonal then  $x^T y = y^T x = 0$  with

$$\|x + y\|^2 \equiv (x + y)^T (x + y), \quad \|x\|^2 \equiv x^T x, \quad \|y\|^2 \equiv y^T y$$

so that

$$\|x + y\|^2 = (x + y)^T (x + y) = \underbrace{x^T x}_{\|x\|^2} + \underbrace{x^T y}_0 + \underbrace{y^T x}_0 + \underbrace{y^T y}_{\|y\|^2} = \|x\|^2 + \|y\|^2.$$