Duration: 50 minutes.

NAME (in ink): **STUDENT NO (in ink):**

PART I: [6] Multiple choice questions. Circle the correct answer in ink.

1. Let $C = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ be a basis of M_{22}. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $[A]_C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, the value of a_{21} is

a) 2 b) 3 c) 8 **d) 9**

2. Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ be a basis of the subspace of H. Let $x = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} \in H$ and $[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, then the value of c_2 is

a) -2 b) 2 c) -1 d) 3

3. Let $T : M_{22} \to P_2$ be defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a-b) + (b-c)x + (c-a)x^2$.

The nullity of T equals to

a) 0 b) 1 **c) 2** d) 3

PART II: [24] Long answer questions. Show all your work.

[7] 1. Let $T : \Re^4 \to \Re^3$ be a linear transformation such that A is the standard matrix of the transformation. The matrices A and B given below are row equivalent.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & 4 & 5 \\ 2 & 4 & 5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i)[2] Find a basis for the kernel of T.

ANS: Solve $AX = 0$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ -1 & -2 & 4 & 5 & 0 \\ 2 & 4 & 5 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$ (Already given as A and B are row equivalent)

$$AX = 0 \Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 \in \Re$$

$\ker(T) = span\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis of $\ker(T)$.

(ii)[2] Find a basis of range of T.

ANS: A basis for range of T is same as a basis for the column space of A.
As columns 1, 3 and 4 of B has pivots, the columns 1, 3, 4 form a basis of range of T.

i.e. $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix},\begin{bmatrix}4\\5\\7\end{bmatrix}\right\}$ form a basis of range of T.

(iii)[3] Explain if the transformation is (i) one-to-one (ii) onto (iii) isomorphism.
ANS:
$Rank(T)=\dim(\Re^4)-\dim(\ker(T))=4-1=3=\dim(\Re^3)$. So T is onto.
$\dim(\ker(T))=nullity(T)=1\neq 0$, T is not one-to-one and not isomorphism.

[17] 2. Let $T: P_2 \to P_3$ be a linear transformation given by

$$T(a+bx+cx^2)=(x+5)(a+bx+cx^2).$$

Let $B=\{1,x,x^2\}$ be a basis of P_2, $C=\{1,x,x^2,x^3\}$ and $D=\{1+x^2, 2+x, x-x^2, x^3\}$ be two bases of P_3 and $v=2-x+x^2$ be a vector in P_2.

(i) [2] Find $[v]_B$, the coordinate vector of v with respect to the basis B.

ANS: $[v]_B=\begin{bmatrix}2\\-1\\1\end{bmatrix}$

(ii)[6] Find the change-of-basis matrix $P_{D\leftarrow C}$.
ANS:
Need to find $[1]_D, [x]_D, [x^2]_D, [x^3]_D$.
Any linear combination of vectors in D is given by
$c_1(1+x^2)+c_2(2+x)+c_3(x-x^2)+c_4x^3=(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3$
So need to solve simultaneously 4 linear systems
$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=1$
$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=x$
$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=x^2$
$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=x^3$

$$\Rightarrow\left[\begin{array}{cccc|cccc}1&2&0&0&1&0&0&0\\0&1&1&0&0&1&0&0\\1&0&-1&0&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]\sim(R_3'=R_3-R_1)\left[\begin{array}{cccc|cccc}1&2&0&0&1&0&0&0\\0&1&1&0&0&1&0&0\\0&-2&-1&0&-1&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]$$

$$\sim(R_3'=R_3+2R_2)\left[\begin{array}{cccc|cccc}1&2&0&0&1&0&0&0\\0&1&1&0&0&1&0&0\\0&0&1&0&-1&2&1&0\\0&0&0&1&0&0&0&1\end{array}\right]\sim(R_2'=R_2-R_3)\left[\begin{array}{cccc|cccc}1&2&0&0&1&0&0&0\\0&1&0&0&1&-1&-1&0\\0&0&1&0&-1&2&1&0\\0&0&0&1&0&0&0&1\end{array}\right]$$

$$\sim(R_1'=R_1-2R_2)\left[\begin{array}{cccc|cccc}1&0&0&0&-1&2&2&0\\0&1&0&0&1&-1&-1&0\\0&0&1&0&-1&2&1&0\\0&0&0&1&0&0&0&1\end{array}\right]$$

$$P_{D\leftarrow C}=\begin{bmatrix}-1&2&2&0\\1&-1&-1&0\\-1&2&1&0\\0&0&0&1\end{bmatrix}$$

(iii)[6] Find $M_{DB}(T)$, the matrix of T corresponding to the ordered bases B and D.

ANS:

Need to find $[T(1)]_D, [T(x)]_D, [T(x^2)]_D$ i.e. $[5+x]_D, [5x+x^2]_D, [5x^2+x^3]_D$.

Any linear combination of vectors in D is given by

$$c_1(1+x^2)+c_2(2+x)+c_3(x-x^2)+c_4x^3=(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3$$

So need to solve simultaneously 3 linear systems

$$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=5+x$$

$$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=5x+x^2$$

$$(c_1+2c_2)+(c_2+c_3)x+(c_1-c_3)x^2+c_4x^3=5x^2+x^3$$

NOTE: see the similarity with solution of (ii)

$$\Rightarrow \left[\begin{array}{cccc|ccc} 1 & 2 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 5 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & -3 & 12 & 10 \\ 0 & 1 & 0 & 0 & 4 & -6 & -5 \\ 0 & 0 & 1 & 0 & -3 & 11 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(Using the exact same row operations in (b)

$$M_{DB}(T)=\begin{bmatrix} -3 & 12 & 10 \\ 4 & -6 & -5 \\ -3 & 11 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv)[3] Use $M_{DB}(T)$ and $[v]_B$ to compute $[T(v)]_D$.

ANS:

$$[T(v)]_D = M_{DB}(T).[v]_B = \begin{bmatrix} -3 & 12 & 10 \\ 4 & -6 & -5 \\ -3 & 11 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 9 \\ -12 \\ 1 \end{bmatrix}$$