tutorial section

COMP1805-BCM Discrete Structures I

Assignment 3

Due Date: March 10, 2016

Time Due: Before 3:00pm in the drop box in HP 3115

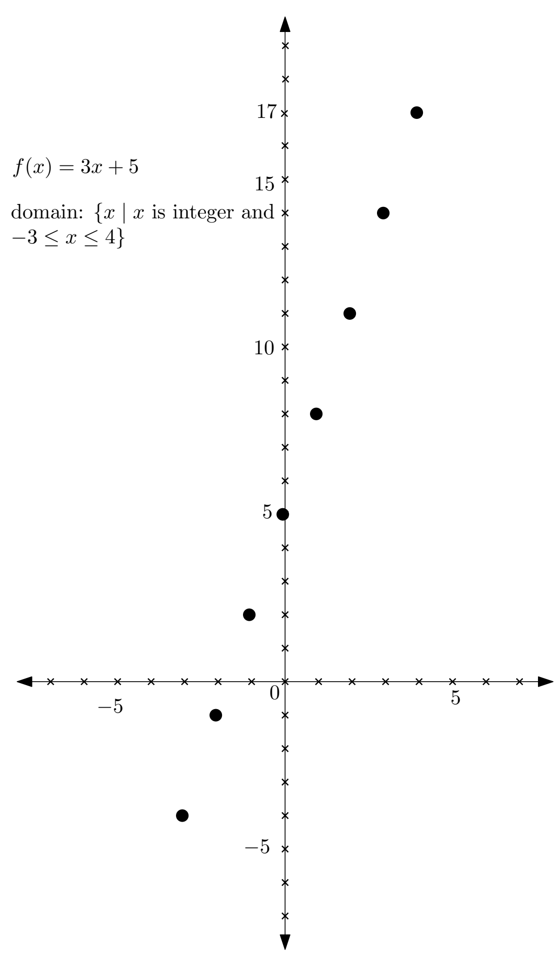
Grade / 40

Student Number ____________________ **Student Name** ____________________________________

- Write down your name and student number on **every** page.
- You must have a cover page that clearly states **your name, student number, course number and tutorial section**. If you do not have a cover page with this information, your assignment **will not be marked.**
- The questions should be answered in order and your assignment sheets must be stapled, otherwise the assignment will not be marked.
- Every part of every question is worth 2 marks. The grading scheme for each question is 2 for correct answer, 1 for an answer that is not completely correct and 0 otherwise.

1. Draw the graph of the function $f(x) = 3x + 5$ where the domain is $\{x \mid x \text{ is an integer and } -3 \leq x \leq 4\}$.

   Solution: See the figure below.



2. Let f and g both be functions from reals to reals. Let $f(x) = 3x - 5$ and $g(x) = x^2 - 4$. Define what is:

   (a) $f \circ g$
   Solution:

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x^2 - 4) \\ &= 3(x^2 - 4) - 5 \end{aligned}$$

   (b) $g \circ f$
   Solution:

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(3x - 5) \\ &= (3x - 5)^2 - 4 \end{aligned}$$

   (c) $(f \circ g) \circ f$
   Solution:

$$\begin{aligned} (f \circ g) \circ f(x) &= f(g(f(x))) \\ &= f(g(3x - 5)) \\ &= f((3x - 5)^2 - 4)) \\ &= 3((3x - 5)^2 - 4) - 5 \end{aligned}$$

3. Compute the closed form for the following (e.g. the closed form for $\sum_{i=1}^{n} i$ is $n(n+1)/2$.):

   (a) $\sum_{i=25}^{n} 5$, where n is a positive integer whose value is at least 25.
   Solution: $\sum_{i=25}^{n} 5 = 5(n - 25 + 1)$

   (b) $\sum_{i=1}^{n} 6(i + 3)$
   Solution:

$$\begin{aligned} \sum_{i=1}^{n} 6(i+3) &= 6\sum_{i=1}^{n}(i+3) \\ &= 6(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 3) \\ &= 6(n(n+1)/2 + 3n) \end{aligned}$$

   (c) $\sum_{i=1}^{n}(3i^2 + 2i - 4)$
   Solution:

$$\begin{aligned} \sum_{i=1}^{n}(3i^2 + 2i - 4) &= \sum_{i=1}^{n} 3i^2 + \sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 4 \\ &= 3\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i - 4\sum_{i=1}^{n} 1 \\ &= 3(n(n+1)(2n+1))/6 + 2(n(n+1)/2) - 4n \end{aligned}$$

   (d) $\sum_{j=33}^{57} 31$
   Solution: $\sum_{j=33}^{57} 31 = 31(57 - 33 + 1)$

   (e) $\sum_{j=1}^{n}(2j^2 - n + 3)$
   Solution:

$$\begin{aligned} \sum_{j=1}^{n}(2j^2 - n + 3) &= \sum_{j=1}^{n} 2j^2 - \sum_{j=1}^{n} n + \sum_{j=1}^{n} 3 \\ &= 2\sum_{j=1}^{n} j^2 - n\sum_{j=1}^{n} 1 + 3\sum_{j=1}^{n} 1 \\ &= 2(n(n+1)(2n+1))/6 - n^2 + 3n \end{aligned}$$

(f) $\sum_{i=1}^{n}\sum_{j=1}^{i}(i+j)$
Solution:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{i}(i+j) &= \sum_{i=1}^{n}(\sum_{j=1}^{i} i + \sum_{j=1}^{i} j) \\
&= \sum_{i=1}^{n}\left(i\sum_{j=1}^{i} 1 + \sum_{j=1}^{i} j\right) \\
&= \sum_{i=1}^{n}(i^2 + i(i+1)/2) \\
&= \sum_{i=1}^{n}(i^2 + i^2/2 + i/2) \\
&= \sum_{i=1}^{n}(3i^2/2 + i/2) \\
&= \sum_{i=1}^{n} 3i^2/2 + \sum_{i=1}^{n} i/2 \\
&= (3/2)\sum_{i=1}^{n} i^2 + (1/2)\sum_{i=1}^{n} i \\
&= (3/2)(n(n+1)(2n+1)/6) + (1/2)(n(n+1)/2)
\end{aligned}$$

(g) $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(3i-1)$
Solution:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(3i-1) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(3i-1)\sum_{k=1}^{n} 1 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(3i-1)n \\
&= \sum_{i=1}^{n}((3i-1)n)\sum_{j=1}^{n} 1 \\
&= \sum_{i=1}^{n}((3i-1)n)n \\
&= \sum_{i=1}^{n}(3i-1)n^2 \\
&= \sum_{i=1}^{n} 3in^2 - n^2 \\
&= \sum_{i=1}^{n} 3in^2 - \sum_{i=1}^{n} n^2 \\
&= 3n^2\sum_{i=1}^{n} i - n^2\sum_{i=1}^{n} 1 \\
&= 3n^2((n)(n+1)/2) - n^3
\end{aligned}$$

(h) $\sum_{i=1}^{n}\sum_{j=i}^{n} 2j$

Solution:

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=i}^{n} 2j &= \sum_{i=1}^{n} 2 \sum_{j=i}^{n} j \\
&= \sum_{i=1}^{n} 2(\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j) \\
&= \sum_{i=1}^{n} 2(n(n+1)/2 - (i-1)(i)/2) \\
&= \sum_{i=1}^{n}(n(n+1) - (i-1)(i)) \\
&= \sum_{i=1}^{n} n(n+1) - \sum_{i=1}^{n}(i-1)(i) \\
&= n(n+1)\sum_{i=1}^{n} 1 - \sum_{i=1}^{n}(i^2 - i) \\
&= n(n+1)n - (\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i) \\
&= n(n+1)n - \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \\
&= n^2(n+1) - n(n+1)(2n+1)/6 + n(n+1)/2
\end{aligned}
$$

4. Determine whether or not the following are true and provide a derivation explaining your answer. The domain of the functions of n below is the positive real numbers. For convenience, you may assume that the logs are in the base of your choice, but you should specify what base you are using in your derivation.

   We provide the definition of O, Ω and Θ below. The domain of all functions is the positive integers. $f(n) \in O(g(n))$ provided that $f(n) \leq cg(n), \forall n \geq d$, for constants $c, d > 0$.
   $f(n) \in O(g(n))$ provided that $\lim_{n\to\infty} f(n)/g(n) \leq c$ for constant $c > 0$.
   $f(n) \in \Omega(g(n))$ provided that $f(n) \geq cg(n), \forall n \geq d$, for constants $c, d > 0$.
   $f(n) \in \Omega(g(n))$ provided that $\lim_{n\to\infty} f(n)/g(n) \geq c$ for constant $c > 0$.
   $f(n)$ is $\Theta(g(n))$ provided that $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

   (a) $(2n-6)^2$ is $\Theta(n^2)$
   Solution:

$$
\begin{aligned}
\lim_{n\to\infty}(2n-6)^2/n^2 &= \lim_{n\to\infty}(4n^2 - 24n + 36)/n^2 \\
&= \lim_{n\to\infty}(4n^2/n^2 - 24n/n^2 + 36/n^2) \\
&= \lim_{n\to\infty}(4 - 24/n + 36/n^2) \to 4
\end{aligned}
$$

   We have that $3 \leq \lim_{n\to\infty}(2n-6)^2/n^2 \leq 5$. Therefore, $(2n-6)^2$ is $\Theta(n^2)$.

   (b) $9n^2 - 7n + 8$ is $O(n^2)$.
   Solution:

$$
\begin{aligned}
9n^2 - 7n + 8 &\leq 9n^2 + 8, \forall n \geq 1 \\
&\leq 9n^2 + 8n^2, \forall n \geq 1 \\
&\leq 17n^2, \forall n \geq 1
\end{aligned}
$$

   Therefore, $9n^2 - 7n + 8$ is $O(n^2)$.

(c) $n^3/27 - 5n^2 \log^2 n$ is $\Theta(n^2)$.
Solution: We will show that $n^3/27 - 5n^2 \log^2 n$ is not $\Theta(n^2)$ by showing that it is $\Omega(n^3)$.
We assume the base of the log is e.

$$\begin{aligned}
\lim_{n\to\infty}(n^3/27 - 5n^2\log^2 n)/n^3 &= \lim_{n\to\infty}((n^3/27n^3) - (5n^2\log^2 n)/n^3) \\
&= \lim_{n\to\infty}(n^3/27n^3) - \lim_{n\to\infty}(5n^2\log^2 n)/n^3 \\
&= \lim_{n\to\infty}(1/27) - \lim_{n\to\infty}(5\log^2 n)/n \\
&= (1/27) - \lim_{n\to\infty}(10\log n)/n \text{ by application of l'Hopital's rule} \\
&= (1/27) - \lim_{n\to\infty} 10/n \text{ by application of l'Hopital's rule} \\
&= 1/27
\end{aligned}$$

Since $\lim_{n\to\infty}(n^3/27 - 5n^2\log^2 n)/n^3 \to 1/27 \geq 1/50$, we conclude that $n^3/27 - 5n^2 \log^2 n$ is $\Omega(n^3)$ which means that it is not $\Theta(n^2)$.

(d) $5n^4 - 3n^3\log^3 n - 3n^2 + 6n\log n$ is $\Omega(n^4)$
Solution:

$$\begin{aligned}
5n^4 - 3n^3\log^3 n - 3n^2 + 6n\log n &\geq 5n^4 - 3n^3 log^3 n - 3n^2, \forall n \geq 1 && (1)\\
&\geq 5n^4 - 3n^4 - 3n^2 && (2)\\
&= 2n^4 - 3n^2 && (3)\\
&= n^4 + (n^4 - 3n^2) && (4)\\
&\geq n^4, \forall n \geq \sqrt{3} && (5)
\end{aligned}$$

To show how we got $\sqrt{3}$ in Line 5 of the derivation, we have to see when $n^4 - 3n^2 \geq 0$.

$$\begin{aligned}
n^4 - 3n^2 &\geq 0 \\
n^4 &\geq 3n^2 \\
n^2 &\geq 3, \text{ divide by } n^2 \\
n &\geq \sqrt{3}
\end{aligned}$$

Therefore, $5n^4 - 3n^3\log^3 n - 3n^2 + 6n\log n$ is $\Omega(n^4)$

(e) $7n^2 - 10n - 14$ is $O(n^3)$.
Solution: $7n^2 - 10n - 14 \leq 7n^2 \leq 7n^3, \forall n \geq 1$. Therefore, $7n^2 - 10n - 14$ is $O(n^3)$.

(f) $3n^2 - n$ is $\Omega(1)$.
Solution:

$$\begin{aligned}
3n^2 - n &\geq 3n^2 - n^2, \forall n \geq 1 \\
&\geq 2n^2, \forall n \geq 1 \\
&\geq 1, \forall n \geq 1
\end{aligned}$$

Since $3n^2 - n \geq c, \forall n \geq d$, with $c = 1, d = 1$, we have that $3n^2 - n$ is $\Omega(1)$.

(g) $9n^{5/2} - 2n^2$ is $O(n^3 \log n)$.
Solution:

$$\begin{aligned}
9n^{5/2} - 2n^2 &\leq 9n^{5/2} \\
&\leq 9n^3,\ \forall n \geq 1 \\
&\leq 9n^3\log n,\ \forall n \geq 2 \text{ base of log is 2.}
\end{aligned}$$

Since we have shown that $9n^{5/2} - 2n^2 \leq 9n^3 \log n,\ \forall n \geq 2$, we conclude that $9n^{5/2} - 2n^2$ is $O(n^3 \log n)$.

(h) $5n/6 - 7$ is $\Omega(n)$.
Solution:

$$\begin{aligned}\lim_{n\to\infty}(5n/6-7)/n &= \lim_{n\to\infty}((5n/6n)-(7/n)) \\ &= \lim_{n\to\infty}(5n/6n) - \lim_{n\to\infty}(7/n) \\ &= \lim_{n\to\infty}(5/6) - \lim_{n\to\infty}(7/n) \\ &\to 5/6\end{aligned}$$

We have $\lim_{n\to\infty}(5n/6-7)/n \to 5/6 \geq 1/2$, we conclude that $5n/6-7$ is $\Omega(n)$.