**MAT 1322 D - Calculus II**
**Midterm Examination II Version 1**

Instructor: Petko Kitanov
University of Ottawa - Department of Mathematics and Statistics
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Solutions

Name:........................................ Student Number:............

# Instructions :

- Please write your name and student number on the indicated area above.
- This is a closed book exam. It contains **7 questions**; there are 50 points in total.
- You can use non-programable and non-graphical calculators but no other aids are permitted.
- Clearly indicate the solution of each problem.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- If you need extra space, use the last page or the back of the pages.
- Time allowed: 80 minutes.

GOOD LUCK!

Final Grade : __________ out of 50

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Grade | | | | | | | |

**Question 1.** [5 points] Express the number $2.\overline{516} = 2.516516516\ldots$ as a ratio of two integers.

$$2.\overline{516} = 2 + 0.\overline{516}$$

$$= 2 + \frac{516}{10^3} + \frac{516}{10^6} + \frac{516}{10^9} + \cdots$$

$$a = \frac{516}{10^3}, \qquad r = \frac{1}{10^3}$$

$$2.\overline{516} = 2 + \frac{516/10^3}{1 - \frac{1}{10^3}} = 2 + \frac{516/10^3}{999/10^3}$$

$$= 2 + \frac{516}{999} = 2 + \frac{172}{333}$$

$$= \frac{838}{333}$$

**Question 2.** [7 points] Use the Integral Test to find the value of $p$ for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent. Justify your answer.

$$f(x) = \frac{1}{x(\ln x)^p}$$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_2^{\infty} \frac{d(\ln x)}{(\ln x)^p} = \int_2^{\infty} (\ln x)^{-1} d(\ln x)$$

$$= \lim_{t \to \infty} \int_2^t (\ln x)^{-1} d(\ln x) = \lim_{t \to \infty} \frac{(\ln x)^{-p+1}}{-p+1} \Big|_2^t$$

$$= \frac{1}{1-p} \left[ \lim_{t \to \infty} (\ln t)^{1-p} - \ln 2 \right]$$

If $p > 1$, then $\lim_{t \to \infty} (\ln t)^{1-p} = 0$

If $p \leq 1$, then $\lim_{t \to \infty} (\ln t)^{1-p} = \infty$

Therefore, for convergence we need $p > 1$.

**Question 3.** [7 points] Use appropriate comparison test to determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{2n^3+n^2}$ (b) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(a) $$\frac{\sqrt{n^4+1}}{2n^3+n^2} > \frac{\sqrt{n^4}}{2n^3+n^3} = \frac{n^2}{3n^3} = \frac{1}{3n}$$

$\sum_{n=1}^{\infty} \frac{1}{3n}$ is divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{2n^3+n^2}$ is divergent.

(b) Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent)

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$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1 > 0$$

Therefore, by Limit Comparison Test, $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ is divergent.

**Question 4.** [ 6 points] Show that the given series is convergent. How many terms of series do we need in order to find the sum to the indicated accuracy.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \qquad (|error| < 0.00005)$$

The given series is convergent, because it is alternating series and $\lim_{n\to\infty} \frac{1}{n^6} = 0$.

$$|R_n| = |s - s_n| \leq b_{n+1}$$

$$b_{n+1} = \frac{1}{(n+1)^6}$$

So, $|R_n| \leq \frac{1}{(n+1)^6} = \frac{5}{100\,000}$

$$(n+1)^6 = 20\,000$$

$$n+1 \approx 5.2$$

$n = 7$ will work.

**Question 5.** [7 points] Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is absolutely convergent, conditionally convergent, or divergent.

The Series $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n}{n^2+4}\right|$

is divergent, because

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n}{n^2+4}\right| = \sum_{n=1}^{\infty}\frac{n}{n^2+4}$$

is divergent, since

$$\lim_{n\to\infty}\frac{n/n^2+4}{1/n} = \lim_{n\to\infty}\frac{n^2}{n^2+4} = 1 > 0$$

(Limit comparison Test)

The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$ is convergent

by Alternating Series Test, since

$$\lim_{n\to\infty}\frac{n}{n^2+4} = 0 \quad \text{and} \quad b_n > b_{n+1}$$

Therefore, the series is NOT Absolutely convergent. It is conditionally convergent.

**Question 6.** [8 points] Determine the interval and the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.

Both ratio and root tests can be used.

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^2+1}\right|} = \lim_{n\to\infty} \frac{|x-2|}{\sqrt[n]{n^2+1}} = |x-2| < 1$$

$$|x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

End Points:

$\underline{x=1}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

convergent (AST)

$\underline{x=3}$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}$$

convergent (p-series)

Therefore, the interval is $[1, 3]$ and $R = 1$

**Question 7.** [10 points] Find a power series representation for the function $\ln(5-x)$. Determine the interval and the radius of convergence.

$$f(x) = \ln(5-x)$$

$$f'(x) = -\frac{1}{5-x} = -\frac{1}{5}\,\frac{1}{1-\frac{x}{5}}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$$

Integrate, to obtain

$$\ln(5-x) = C - \sum_{n=1}^{\infty}\int\frac{x^n}{5^{n+1}}\,dx = C - \sum_{n=1}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}}$$

To determine $C$, let $x=0$

$\ln 5 = C$, So, we have

$$\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{x}{5}\right)^{n+1}$$

$$\left|\frac{x}{5}\right| < 1 \Rightarrow -5 < x < 5$$

If $x=-5$, then $\ln 5 - \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ is

convergent by AST, for $\lim_{n\to\infty}\frac{1}{n+1} = 0$

If $x=5$, the function is not defined.

Therefore, the interval of convergence is $[-5, 5)$ and $R=5$.