CVG2140 – Solutions to Assignment No. 6 (Deflections)

Problem 1. Determine the equations of the elastic curve for the beam illustrated in Fig. 1 using the *x* coordinate as shown. Specify the slope at *A* and the maximum displacement of the shaft. *EI* is constant.

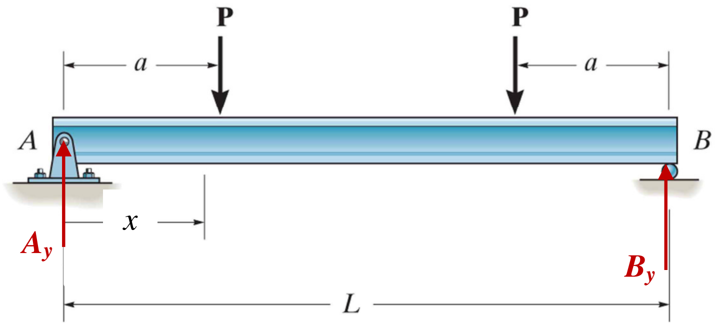

Fig. 1

Establishing equilibrium:

$$\left.\begin{aligned}\sum F_y &= A_y - 2P + B_y = 0\\ \sum M_A &= -(P\times a) - \left[P\times(L-a)\right] + \left(B_y \times L\right) = 0\end{aligned}\right\} \Rightarrow \begin{cases}A_y = P\\ B_y = P\end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = P\langle x-0\rangle - P\langle x-a\rangle - P\langle x-(L-a)\rangle$$

$$EI\frac{d^2v}{dx^2} = M(x) = Px - P\langle x-a\rangle - P\langle x-(L-a)\rangle$$

$$EI\frac{dv}{dx} = \frac{P}{2}x^2 - \frac{P}{2}\langle x-a\rangle^2 - \frac{P}{2}\langle x-(L-a)\rangle^2 + C_1 \qquad \text{(Eq. 1)}$$

$$EIv = \frac{P}{6}x^3 - \frac{P}{6}\langle x-a\rangle^3 - \frac{P}{6}\langle x-(L-a)\rangle^3 + C_1x + C_2 \qquad \text{(Eq. 2)}$$

The boundary conditions of the problem are:

$$x = 0 \Rightarrow v = 0$$

$$x = L \Rightarrow v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$x = 0 \Rightarrow v = 0 = C_2$$

$$x = L \Rightarrow v = 0 = \frac{P}{6}L^3 - \frac{P}{6}(L-a)^3 - \frac{P}{6}a^3 + C_1L \Rightarrow C_1 = -\frac{Pa}{2}(L-a)$$

The equations for the slope and deflection are therefore given by:

$$EI\frac{dv}{dx} = \frac{P}{2}x^2 - \frac{P}{2}\langle x-a\rangle^2 - \frac{P}{2}\langle x-(L-a)\rangle^2 - \frac{Pa}{2}(L-a) \qquad \text{(Eq. 3)}$$

$$EIv = \frac{P}{6}x^3 - \frac{P}{6}\langle x-a\rangle^3 - \frac{P}{6}\langle x-(L-a)\rangle^3 - \frac{Pax}{2}(L-a) \qquad \text{(Eq. 4)}$$

The slope at A is calculated from Eq. 3 for $x = 0$:

$$\theta_A = \frac{1}{EI}\left[\frac{P}{2}(0)^2 - 0 - 0 - \frac{Pa}{2}(L-a)\right] = -\frac{Pa}{2EI}(L-a) \qquad \text{(rotation is clockwise)}$$

Because of symmetry in loading and boundary conditions, the maximum deflection occurs at $x = L/2$. Using Eq. 4:

$$v_{\max} = \frac{1}{EI}\left[\frac{P}{6}\left(\frac{L}{2}\right)^3 - \frac{P}{6}\left(\frac{L}{2}-a\right)^3 - 0 - \frac{PaL}{4}(L-a)\right] = \frac{Pa}{24EI}\left(4a^2 - 3L^2\right)$$

Problem 2. Determine the equation of the elastic curve, the slope at A, and the deflection at B of the simply supported beam shown in Fig. 2. EI is constant.

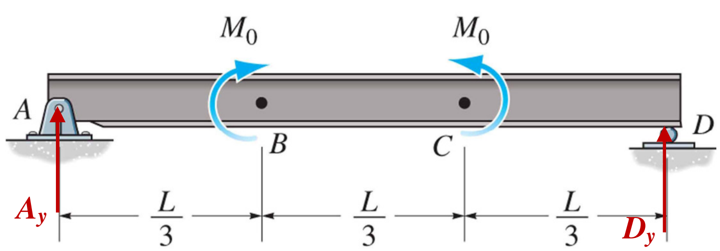

Fig. 2

Establishing equilibrium:

$$\left.\begin{aligned}\sum F_y &= A_y + D_y = 0\\ \sum M_A &= -M_o + M_o + (D_y \times L) = 0\end{aligned}\right\} \Rightarrow \begin{cases}A_y = 0\\ D_y = 0\end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = M_o\left\langle x - \frac{L}{3}\right\rangle^0 - M_o\left\langle x - \frac{2L}{3}\right\rangle^0$$

$$EI\frac{d^2v}{dx^2} = M(x) = M_o\left\langle x - \frac{L}{3}\right\rangle^0 - M_o\left\langle x - \frac{2L}{3}\right\rangle^0$$

$$EI\frac{dv}{dx} = M_o\left\langle x - \frac{L}{3}\right\rangle - M_o\left\langle x - \frac{2L}{3}\right\rangle + C_1 \qquad \text{(Eq. 1)}$$

$$EIv = \frac{M_o}{2}\left\langle x - \frac{L}{3}\right\rangle^2 - \frac{M_o}{2}\left\langle x - \frac{2L}{3}\right\rangle^2 + C_1x + C_2 \qquad \text{(Eq. 2)}$$

The boundary conditions of the problem are:

$$x = 0 \quad \Rightarrow \quad v = 0$$

$$x = L \quad \Rightarrow \quad v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$x = 0 \quad \Rightarrow \quad v = 0 = C_2$$

$$x = L \quad \Rightarrow \quad v = 0 = \frac{M_o}{2}\left(\frac{2L}{3}\right)^2 - \frac{M_o}{2}\left(\frac{L}{3}\right)^2 + C_1L \quad \Rightarrow \quad C_1 = -\frac{M_oL}{6}$$

The equations for the slope and deflection are therefore given by:

$$EI\frac{dv}{dx} = M_o\left\langle x-\frac{L}{3}\right\rangle - M_o\left\langle x-\frac{2L}{3}\right\rangle - \frac{M_oL}{6} \quad \text{(Eq. 3)}$$

$$EIv = \frac{M_o}{2}\left\langle x-\frac{L}{3}\right\rangle^2 - \frac{M_o}{2}\left\langle x-\frac{2L}{3}\right\rangle^2 - \frac{M_oL}{6}x \quad \text{(Eq. 4)}$$

The slope at A is calculated from Eq. 3 for $x = 0$:

$$\theta_A = -\frac{M_oL}{6EI}$$

The deflection at B is calculated from Eq. 4 for $x = L/3$:

$$v_B = \frac{1}{EI}\left(-\frac{M_oL}{6}\cdot\frac{L}{3}\right) = -\frac{M_oL^2}{18EI}$$

Problem 3. Determine the maximum deflection of the simply supported beam shown in Fig. 3. E = 200 GPa and $I = 65.0\times10^6$ mm^4.

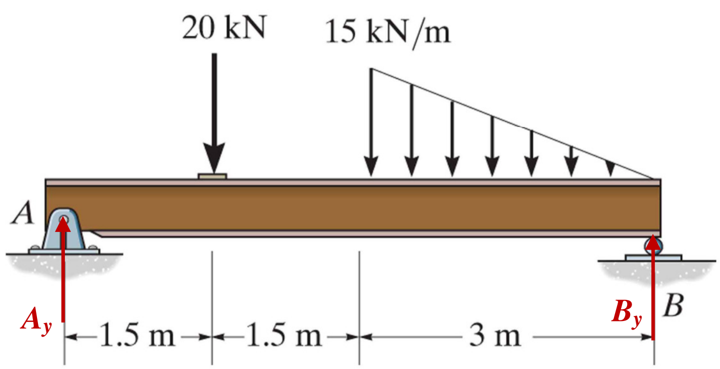

Fig. 3

Establishing equilibrium:

$$\left.\begin{aligned}\sum F_y &= A_y - 20 - \left(\frac{1}{2}\times15\times3\right) + B_y = 0\\ \sum M_A &= -(20\times1.5) - \left(\frac{1}{2}\times15\times3\right)(3+1) + (B_y\times6) = 0\end{aligned}\right\} \Rightarrow \begin{cases}A_y = 22.5 \text{ kN}\\ B_y = 20 \text{ kN}\end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = 22.5\langle x-0\rangle - 20\langle x-1.5\rangle - \frac{15}{2}\langle x-3\rangle^2 + \frac{15}{18}\langle x-3\rangle^3 - \frac{15}{18}\langle x-6\rangle^3$$

$$EI\frac{d^2v}{dx^2} = M(x) = 22.5x - 20\langle x-1.5\rangle - \frac{15}{2}\langle x-3\rangle^2 + \frac{15}{18}\langle x-3\rangle^3 - \frac{15}{18}\langle x-6\rangle^3$$

$$EI\frac{dv}{dx} = \frac{22.5}{2}x^2 - \frac{20}{2}\langle x-1.5\rangle^2 - \frac{15}{6}\langle x-3\rangle^3 + \frac{15}{72}\langle x-3\rangle^4 - \frac{15}{72}\langle x-6\rangle^4 + C_1 \quad \text{(Eq. 1)}$$

$$EIv = \frac{22.5}{6}x^3 - \frac{20}{6}\langle x-1.5\rangle^3 - \frac{15}{24}\langle x-3\rangle^4 + \frac{15}{360}\langle x-3\rangle^5 - \frac{15}{360}\langle x-6\rangle^5 + C_1x + C_2 \quad \text{(Eq. 2)}$$

The boundary conditions of the problem are:

$$x = 0 \quad \Rightarrow \quad v = 0$$

$$x = 6 \quad \Rightarrow \quad v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$x = 0 \quad \Rightarrow \quad v = 0 = C_2$$

$$x = 6 \quad \Rightarrow \quad v = 0 = \left(22.5 \times 6^2\right) - \left(\frac{20}{6} \times 4.5^3\right) - \left(\frac{15}{24} \times 3^4\right) + \left(\frac{15}{360} \times 3^5\right) + 6C_1 \quad \Rightarrow$$

$$C_1 = -77.625$$

The equations for the slope and deflection are therefore given by:

$$EI\frac{dv}{dx} = 11.25x^2 - 10\langle x-1.5\rangle^2 - 2.5\langle x-3\rangle^3 + 0.2083\langle x-3\rangle^4 - 0.2083\langle x-6\rangle^4 - 77.625 \qquad \text{(Eq. 3)}$$

$$EIv = 3.75x^3 - 3.33\langle x-1.5\rangle^3 - 0.625\langle x-3\rangle^4 + 0.0417\langle x-3\rangle^5 - 0.0417\langle x-6\rangle^5 - 77.625x \qquad \text{(Eq. 4)}$$

The maximum deflection occurs when the slope is zero. Assuming that $\frac{dv}{dx} = 0$ occurs in the region 1.5 m < x < 3 m, then:

$$\frac{dv}{dx} = 0 = \frac{1}{EI}\left[11.25x^2 - 10(x-1.5)^2 - 0 + 0 - 0 - 77.625\right]$$

Solving for the root 1.5 m < x < 3 m, x = 2.97 m (the solution is within assumed bounds, therefore it is O.K.)

The maximum deflection is calculated from Eq. 4 for x = 2.97 m:

$$v_{\max} = \frac{1}{200\times10^6 \times 65\times10^{-6}}\left[3.75(2.97)^3 - 3.33(1.47)^3 - 77.625(2.97)\right] = \underline{-0.01099\text{ m}}$$

Problem 4. Determine the displacement at C and the slope at A of the beam shown in Fig. 4.

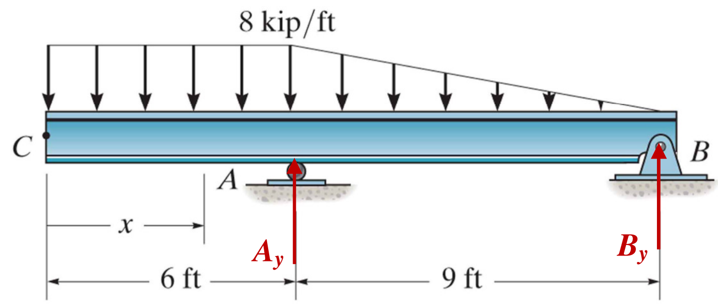

Fig. 4

Establishing equilibrium:

$$\left.\begin{array}{l}\sum F_y = A_y - (8\times6) - \left(\frac{1}{2}\times8\times9\right) + B_y = 0 \\ \sum M_A = (8\times6\times3) - \left(\frac{1}{2}\times8\times9\right)(3) + (B_y\times9) = 0\end{array}\right\} \Rightarrow \begin{cases} A_y = 88\text{ kip} \\ B_y = -4\text{ kip}\end{cases}$$

Using discontinuity functions, the moment equation is written as:

$$M(x) = -\frac{8}{2}\langle x-0\rangle^2 + \frac{8}{2}\langle x-6\rangle^2 + 88\langle x-6\rangle - \frac{8}{2}\langle x-6\rangle^2 + \frac{8}{6\times 9}\langle x-6\rangle^3 - \frac{8}{6\times 9}\langle x-15\rangle^3$$
$$-4\langle x-15\rangle$$

$$EI\frac{d^2v}{dx^2} = M(x) = -4x^2 + 88\langle x-6\rangle + \frac{4}{27}\langle x-6\rangle^3$$

$$EI\frac{dv}{dx} = -\frac{4}{3}x^3 + \frac{88}{2}\langle x-6\rangle^2 + \frac{4}{27\times 4}\langle x-6\rangle^4 + C_1 \qquad \text{(Eq. 1)}$$

$$EIv = -\frac{4}{12}x^4 + \frac{88}{6}\langle x-6\rangle^3 + \frac{4}{27\times 20}\langle x-6\rangle^5 + C_1x + C_2 \qquad \text{(Eq. 2)}$$

The boundary conditions of the problem are:

$$x = 6 \quad \Rightarrow \quad v = 0$$
$$x = 15 \quad \Rightarrow \quad v = 0$$

The constants of integration C_1 and C_2 are then determined by applying the boundary conditions to Eq. 2:

$$\left.\begin{aligned} x = 6 \quad &\Rightarrow \quad v = 0 = -\frac{4}{12}6^4 + 6C_1 + C_2 \\ x = 15 \quad &\Rightarrow \quad v = 0 = -\frac{4}{12}15^4 + \frac{88}{6}9^3 + \frac{4}{27\times 20}9^5 + 15C_1 + C_2 \end{aligned}\right\} \Rightarrow \begin{cases} C_1 = 590.4 \\ C_2 = -3110.4 \end{cases}$$

The equations for the slope and deflection are therefore given by:

$$EI\frac{dv}{dx} = -\frac{4}{3}x^3 + \frac{88}{2}\langle x-6\rangle^2 + \frac{4}{27\times 4}\langle x-6\rangle^4 + 590.4 \qquad \text{(Eq. 3)}$$

$$EIv = -\frac{4}{12}x^4 + \frac{88}{6}\langle x-6\rangle^3 + \frac{4}{27\times 20}\langle x-6\rangle^5 + 590.4x - 3310.4 \qquad \text{(Eq. 4)}$$

The deflection at C is calculated from Eq. 4 for $x = 0$:

$$v_C = \frac{1}{EI}\left[-\frac{4}{12}(0)^4 + 0 + 0 + 590.4(0) - 3310.4\right] = -\frac{3310.4 \text{ kip}\cdot\text{ft}^3}{EI}$$

The slope at A is calculated from Eq. 3 for $x = 6$:

$$\theta_A = \frac{1}{EI}\left[-\frac{4}{3}6^3 + 0 + 0 + 590.4\right] = \frac{302.4 \text{ kip}\cdot\text{ft}^2}{EI}$$