CVG2140 – Solutions to Assignment No. 3 (Centroids and Moments of Inertia)

Problem 1. Calculate the position of the centroid of the shape shown in Fig. 1.

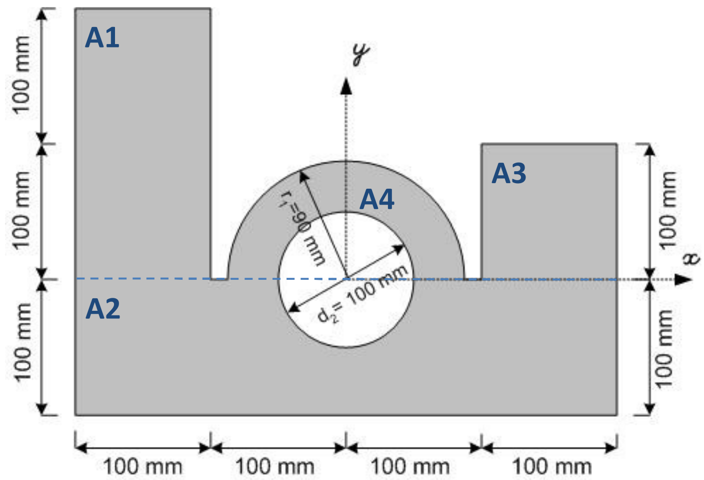


Fig. 1

		A (mm^2)	$\bar{x}_i$ (mm)	$\bar{y}_i$ (mm)	$\bar{x}_i \cdot A_i$ (mm^3)	$\bar{y}_i \cdot A_i$ (mm^3)
A1	Vertical rectangle	20,000	-150	100	-3,000,000	2,000,000
A2	Horizontal rectangle	40,000	0	-50	0	-2,000,000
A3	Square	10,000	150	50	1,500,000	500,000
A4	Semicircle	12,723.5	0	38.197	0	486,000
A5	Empty circle	7,853.98	0	0	0	0
Σ		74,869.5			-1,500,000	986,000

$$\bar{x} = \frac{\sum \bar{x}_i \cdot A_i}{\sum A_i} = -20.0 \quad \text{mm}$$

$$\bar{y} = \frac{\sum \bar{y}_i \cdot A_i}{\sum A_i} = 13.2 \quad \text{mm}$$

Problem 2. Calculate the moments of inertia I_x, I_y, J_o, I_{xy} of the section shown in Fig. 2 with respect to a coordinate system located at the centroid. (Dimensions are in mm).

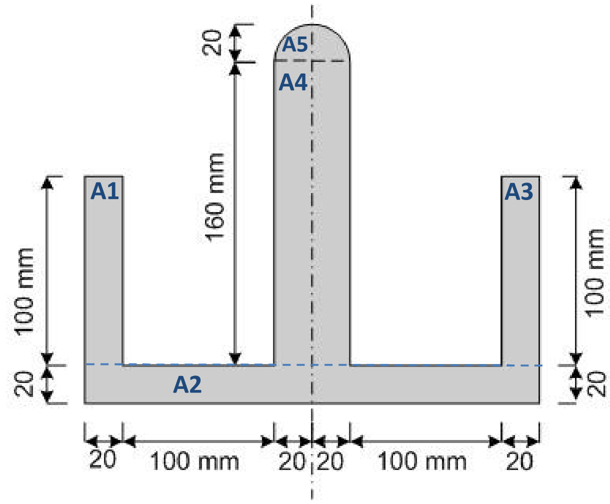


Fig. 2

(1) Centroid

		A (mm²)	$\bar{x}_i$ (mm)	$\bar{y}_i$ (mm)	$\bar{x}_i \cdot A_i$ (mm³)	$\bar{y}_i \cdot A_i$ (mm³)
A1	Vertical rectangle	2,000	-140	70	-280,000	140,000
A2	Horizontal rectangle	5,600	0	10	0	56,000
A3	Vertical rectangle	2,000	140	70	280,000	140,000
A4	Middle rectangle	6,400.00	0	100	0	640,000
A5	Semicircle	628.32	0	188.49	0	118,432
Σ		16,628.32			0.00	1,094,432.04

$$\bar{x} = \frac{\sum \bar{x}_i \cdot A_i}{\sum A_i} = 0 \text{ mm (symmetric with respect to } Y_c\text{)}$$

$$\bar{y} = \frac{\sum \bar{y}_i \cdot A_i}{\sum A_i} = 65.8 \text{ mm (from the bottom)}$$

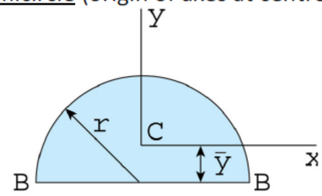
(2) Moments of inertia

- Note that the parallel axis theorem always refers to the centroidal axis of each area.
- The moments of inertia of each area with respect to their own centroidal axes are denoted here by I'_{xc} and I'_{yc}, respectively.
- For area A5, top semicircle, the value given for I'_{xc} is obtained from the formulas in the diagram below, i.e.,

$$I_{BB} = I'_{xc} + Ad^2 \quad \Rightarrow$$

$$I'_{xc} = I_{BB} - Ad^2 = \frac{\pi r^4}{8} - \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right)^2 = 62{,}831.85 - (628.32)(8.49)^2 = 17{,}542.48 \text{ mm}^4$$

Semicircle (origin of axes at centroid)



$$A = \frac{\pi r^2}{2}, \quad \bar{y} = \frac{4r}{3\pi}$$

$$I_x = \frac{(9\pi^2 - 64) r^4}{72\pi} \approx 0.1098 r^4, \quad I_y = \frac{\pi r^4}{8}, \quad I_{xy} = 0$$

$$I_{BB} = \frac{\pi r^4}{8}$$

	A (mm^2)	I'_{xc} (mm^4)	I'_{yc} (mm^4)	d_x (mm)	d_y (mm)	Ad_x^2 (mm^4)	Ad_y^2 (mm^4)	$I_{xc} = I'_{xc} + Ad_x^2$ (mm^4)	$I_{yc} = I'_{yc} + Ad_y^2$ (mm^4)	$J_c = I_{xc} + I_{yc}$ (mm^4)
A1	2,000	1,666,666	66,666	4.2	-130	35,280	33,800,000	1,701,947	33,866,667	35,568,613
A2	5,600	186,666	36,586,666	-55.8	0	17,436,384	0	17,623,051	36,586,667	54,209,717
A3	2,000	1,666,666	66,666	4.2	130	35,280	33,800,000	1,701,947	33,866,667	35,568,613
A4	6,400	13,653,333	853,333	34.2	0	7,485,696	0	21,139,029	853,333	21,992,363
A5	628	17,542	62,831	122.7	0	9,457,998	0	9,475,540	62,832	9,538,372
Σ	16,628							**51,641,514**	**105,236,165**	**156,877,679**

From the above table the moments of inertia are given by:

$I_x = I_{xc} = 51{,}641{,}514 \text{ mm}^4$

$I_y = I_{yc} = 105{,}236{,}165 \text{ mm}^4$

$J_o = J_c = 156{,}877{,}679 \text{ mm}^4$

(3) Product of inertia

All the areas have a vertical axis of symmetry; therefore, the product of inertia with respect to the centroidal axes is zero. Note that the product of inertia with respect to an axis of symmetry is zero. This is also applicable for the composite area, since it is symmetric with respect to the vertical centroidal axis *y*-*y*. This can be proved by calculating it as shown in the following table, although it is not necessary for the solution of this problem.

	A (mm^2)	I'_{xy} (mm^4)	d_x (mm)	d_y (mm)	$Ad_x d_y$ (mm^4)	$I_{xy} = I'_{xy} + Ad_x d_y$ (mm^4)
A1	2,000	0	4.2	-130	-1,092,000	-1,092,000
A2	5,600	0	-55.8	0	0	0
A3	2,000	0	4.2	130	1,092,000	1,092,000
A4	6,400	0	34.2	0	0	0
A5	628	0	122.69	0	0	0
Σ	16,628					**0**

Problem 3. For a coordinate system located at the centroid, calculate the maximum and minimum values of the moments of inertia I_{max} and I_{min} (obtained by rotating the axes) and the corresponding angles of the section shown in Fig. 3. (Dimensions are in mm).

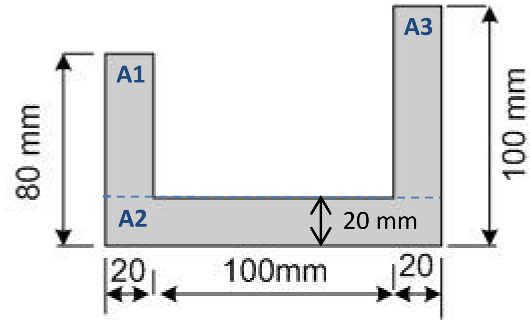


Fig. 3

(1) Centroid

		A (mm²)	$\bar{x}_i$ (mm)	$\bar{y}_i$ (mm)	$\bar{x}_i \cdot A_i$ (mm³)	$\bar{y}_i \cdot A_i$ (mm³)
A1	Vertical rectangle	1,200	10	50	12,000	60,000
A2	Horizontal rectangle	2,800	70	10	196,000	28,000
A3	Vertical rectangle	1,600	130	60	208,000	96,000
Σ		5,600.00			416,000.00	184,000.00

$$\bar{x} = \frac{\sum \bar{x}_i \cdot A_i}{\sum A_i} = 74.3 \text{ mm (from the left edge)}$$

$$\bar{y} = \frac{\sum \bar{y}_i \cdot A_i}{\sum A_i} = 32.8 \text{ mm (from the bottom)}$$

(2) Moments of inertia with respect to the centroid

	A (mm²)	I'_{xc} (mm⁴)	I'_{yc} (mm⁴)	d_x (mm)	d_y (mm)	Ad_x^2 (mm⁴)	Ad_y^2 (mm⁴)	$I_{xc} = I'_{xc} + Ad_x^2$ (mm⁴)	$I_{yc} = I'_{yc} + Ad_y^2$ (mm⁴)
A1	1,200	360000	40000	17.2	-64.3	355,008	4,961,388	715,008	5,001,388
A2	2,800	93333.33	4573333.33	-22.8	-4.3	1,455,552	51,772	1,548,885	4,625,105
A3	1,600	853333.33	53333.33	27.2	55.7	1,183,744	4,963,984	2,037,077	5,017,317
Σ	5,600							**4,300,971**	**14,643,811**

From the above table the moments of inertia are given by:

$I_x = I_{xc} = 4{,}300{,}971 \text{ mm}^4$

$I_y = I_{yc} = 14{,}643{,}811 \text{ mm}^4$

(4) Product of inertia with respect to the centroid

	A (mm^2)	I'_{xy} (mm^4)	d_x (mm)	d_y (mm)	Ad_xd_y (mm^4)	$I_{xy} = I'_{xy} + Ad_xd_y$ (mm^4)
A1	1,200	0	17.2	-64.3	-1,327,152	-1,327,152
A2	2,800	0	-22.8	-4.3	274,512	274,512
A3	1,600	0	27.2	55.7	2,424,064	2,424,064
Σ	5,600.00					**1,371,424**

$I_{xcyy} = I_{xy} = 1{,}371{,}424\ \text{mm}^4$

(5) Principal moments of inertia and corresponding directions

$$\tan 2\theta = -\frac{2I_{xcyc}}{I_{xc} - I_{yc}} = -\frac{2\times 1{,}371{,}424}{4{,}300{,}971 - 14{,}643{,}811} \Rightarrow \begin{cases} \theta_2 = 7.4^\circ \\ \theta_1 = 97.4^\circ \end{cases}$$

$$\begin{aligned} I_{x'} &= \frac{I_{xc}+I_{yc}}{2} + \frac{I_{xc}-I_{yc}}{2}\cos(2\times 7.4) - I_{xcyc}\sin(2\times 7.4) \\ &= \frac{4{,}300{,}971+14{,}643{,}811}{2} + \frac{4{,}300{,}971-14{,}643{,}811}{2}\cos(2\times 7.4) - \left[1{,}371{,}424\times\sin(2\times 7.4)\right] \\ &= 4{,}122{,}217\ \text{mm}^4 = I_{\min} \end{aligned}$$

$$\begin{aligned} I_{y'} &= \frac{I_{xc}+I_{yc}}{2} + \frac{I_{xc}-I_{yc}}{2}\cos(2\times 97.4) - I_{xcyc}\sin(2\times 97.4) \\ &= \frac{4{,}300{,}971+14{,}643{,}811}{2} + \frac{4{,}300{,}971-14{,}643{,}811}{2}\cos(2\times 97.4) - \left[1{,}371{,}424\times\sin(2\times 97.4)\right] \\ &= 14{,}822{,}565\ \text{mm}^4 = I_{\max} \end{aligned}$$