MAT 1322B, Winter 2005 Professor: Venanzio Capretta

MIDTERM TEST 3, version A

Solutions

1. [2 points, 8.6 #8] Find a power series representation of the function:

$$f(x) = \frac{3x}{1-2x^2}$$

A. $\sum_{n=0}^{\infty} (\frac{3}{2})^n x^{2n}$ *B*. $\sum_{n=0}^{\infty} 6^n x^{n+1}$ *C*. $\sum_{n=0}^{\infty} \frac{2}{3^n} x^{2n+2}$

D. $\sum_{n=0}^{\infty} 3 \cdot 2^n x^{2n+1}$ *E*. $\sum_{n=0}^{\infty} \frac{3}{2^n} x^{2n-1}$ *F*. $\sum_{n=0}^{\infty} 2 \cdot 3^n x^{2n+1}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-2x^2} = \sum_{n=0}^{\infty} (2x^2)^n = \sum_{n=0}^{\infty} 2^n x^{2n}$$

$$\frac{3x}{1-2x^2} = \sum_{n=0}^{\infty} 3x \cdot 2^n x^{2n} = \sum_{n=0}^{\infty} 3 \cdot 2^n x^{2n+1}$$

2. [2 points, 8.8 #13] Let $f(x) = \sqrt[3]{1+x^2}$. Evaluate $f^{(6)}(0)$.
[Hint: Use the binomial series to find the Maclaurin series of $f(x)$.]

A. 400 **B.** $\frac{400}{9}$ C. 2 D. -90 E. $\frac{105}{243}$ F. $-\frac{800}{9}$

$$f(x) = \sqrt[3]{1+x^2} = (1+x^2)^{1/3}$$

By the Binomial Series we have:

$$(1+x)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^n$$

So we have:

$$f(x) = (1+x^2)^{1/3} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^{2n}$$

But we also have, by the Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

The coefficient of the term $x^6 = x^{2 \cdot 3}$ from the two formulas must be the same:

$$\frac{f^{(6)}(0)}{6!} = \binom{1/3}{3}$$

$$f^{(6)}(0) = 6! \binom{1/3}{3} = 6! \frac{1/3(1/3-1)(1/3-2)}{3!} = \frac{400}{9}$$

3. [2 points, 11.3 #55] Let $f(x, y, z) = xe^{2z}\sin 3y$. Determine the partial derivative:

$$\frac{\partial^3 f}{\partial x \partial y \partial z}.$$

$\mathbf{A}.\, 6e^{2z}\cos 3y \quad B.\, x^3\cos 3y \quad C.\, 5e^{2z}\cos 3y$

$D.\, 5e^{3z}\cos 2y \quad E. -6e^{3z}\sin 2y \quad F. -xe^{3z}\sin 2y$

$$\frac{\partial f}{\partial z} = 2xe^{2z}\sin 3y$$
$$\frac{\partial^2 f}{\partial y \partial z} = 6xe^{2z}\cos 3y$$
$$\frac{\partial^3 f}{\partial x \partial y \partial z} = 6e^{2z}\cos 3y$$

4. [2 points, 8.7#23, 25] Find the Maclaurin series for the function:

$$f(x) = \int_0^x t^2 e^{3t} dt$$

[Hint: First find the Maclaurin series for $g(t) = t^2 e^{3t}$.]

$A.\ \sum_{n=0}^{\infty} \frac{3^n}{n!(n+4)} x^{n+3}$ $\quad B.\ \sum_{n=0}^{\infty} \frac{2^n}{n!(n+3)} x^{n+4}$ $\quad C.\ \sum_{n=0}^{\infty} \frac{2^n}{n!(n+4)} x^{n+4}$

$D.\ \sum_{n=0}^{\infty} \frac{3^n}{n!(n+3)} x^{n+2}$ $\quad E.\ \sum_{n=0}^{\infty} \frac{2^n}{n!(n+4)} x^{n+3}$ $\quad \mathbf{F}.\ \sum_{n=0}^{\infty} \frac{3^n}{n!(n+3)} x^{n+3}$

$$\begin{aligned}
e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
e^{3t} &= \sum_{n=0}^{\infty} \frac{(3t)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} t^n \\
t^2 e^{3t} &= \sum_{n=0}^{\infty} \frac{3^n}{n!} t^{n+2} \\
\int_0^x t^2 e^{3t} dt &= \int_0^x \sum_{n=0}^{\infty} \frac{3^n}{n!} t^{n+2} dt = \sum_{n=0}^{\infty} \frac{3^n}{n!} \int_0^x t^{n+2} dt \\
&= \sum_{n=0}^{\infty} \frac{3^n}{n!} \frac{x^{n+3}}{n+3} = \sum_{n=0}^{\infty} \frac{3^n}{n!(n+3)} x^{n+3}
\end{aligned}$$

5. [2 points, 11.3 Lecture] Consider the wave equation:

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

Which of the following is a solution for every value of a?

$A.\ z = \ln(ax)\sin y$	$B.\ z = e^{ay}\cos x$	$C.\ z = a\sin x\cos y$
$D.\ z = e^{x-ay}$	**E.** $z = e^{ax-y}$	$F.\ z = a^2 e^{x+y}\cos(x+y)$

The only correct solution is: $z = e^{ax-y}$.

$$\frac{\partial z}{\partial x} = ae^{ax-y} \qquad \frac{\partial z}{\partial y} = -e^{ax-y}$$

$$\frac{\partial^2 z}{\partial x^2} = a^2 e^{ax-y} \qquad \frac{\partial^2 z}{\partial y^2} = e^{ax-y}$$

$$\implies \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

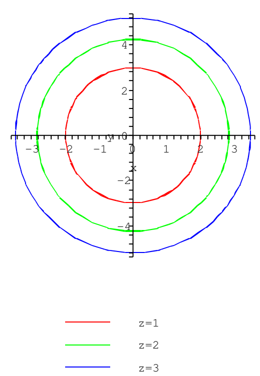
None of the other functions satisfies the equation.

6. [4 points, 11.1 #23] Consider the function:

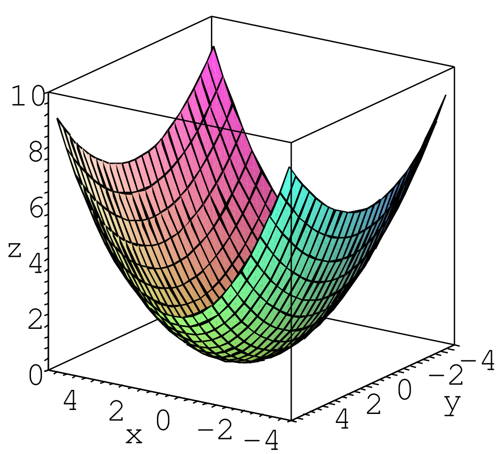
$$f(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$$

(a)[2 points]. Sketch the contour map of the function, using at least 3 level curves (with labels).
We choose the level curves for $z = 1, z = 2, z = 3$:

$$1 = \frac{x^2}{4} + \frac{y^2}{9} \quad \text{ellipse with semiaxes 2 and 3}$$
$$2 = \frac{x^2}{4} + \frac{y^2}{9} \quad \text{ellipse with semiaxes } 2\sqrt{2} \text{ and } 3\sqrt{2}$$
$$3 = \frac{x^2}{4} + \frac{y^2}{9} \quad \text{ellipse with semiaxes } 2\sqrt{3} \text{ and } 3\sqrt{3}$$



(b)[2 points]. Sketch the 3D-graph of the function.



7. [4 points, 11.3 Lecture] Consider the following function of two variables

$$f(x, y) = xye^{2x} + xe^{3xy}$$

(a)[2 points] Find the partial derivatives f_x and f_y at the point $(1, 0)$.

$$f_x = ye^{2x} + 2xye^{2x} + e^{3xy} + 3xye^{3xy} \qquad f_x(1,0) = 1$$
$$f_y = xe^{2x} + 3x^2e^{3xy} \qquad f_y(1,0) = e^2 + 3$$

(b)[2 points] Find the second partial derivatives f_{xx}, f_{yy}, f_{xy}.

$$f_{xx} = 4ye^{2x} + 4xye^{2x} + 6ye^{3xy} + 9xy^2e^{3xy}$$
$$f_{xy} = e^{2x} + 2xe^{2x} + 6xe^{3xy} + 9x^2ye^{3xy}$$
$$f_{yy} = 9x^3e^{3xy}$$

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MIDTERM TEST 3, version B

Solutions

1. [2 points, 8.6 #8] Find a power series representation of the function:

$$f(x) = \frac{2x}{1-3x^2}$$

A. $\sum_{n=0}^{\infty}(\frac{3}{2})^n x^{2n}$ *B.* $\sum_{n=0}^{\infty} 6^n x^{n+1}$ *C.* $\sum_{n=0}^{\infty} \frac{2}{3^n} x^{2n+2}$

D. $\sum_{n=0}^{\infty} 3 \cdot 2^n x^{2n+1}$ *E.* $\sum_{n=0}^{\infty} \frac{3}{2^n} x^{2n-1}$ **F.** $\sum_{n=0}^{\infty} 2 \cdot 3^n x^{2n+1}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-3x^2} = \sum_{n=0}^{\infty} (3x^2)^n = \sum_{n=0}^{\infty} 3^n x^{2n}$$

$$\frac{2x}{1-3x^2} = \sum_{n=0}^{\infty} 2x \cdot 3^n x^{2n} = \sum_{n=0}^{\infty} 2 \cdot 3^n x^{2n+1}$$

2. [2 points, 8.8 #13] Let $f(x) = \sqrt{1+x^3}$. Evaluate $f^{(6)}(0)$.
[Hint: Use the binomial series to find the Maclaurin series of $f(x)$]

A. 400 B. $\frac{400}{9}$ C. 2 **D.** -90 E. $\frac{105}{243}$ F. $-\frac{800}{9}$

$$f(x) = \sqrt{1+x^3} = (1+x^3)^{1/2}$$

By the Binomial Series we have:

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So we have:

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But we also have, by the Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

The coefficient of the term $x^6 = x^{3\cdot 2}$ from the two formulas must be the same:

$$\frac{f^{(6)}(0)}{6!} = \binom{1/2}{2}$$

$$f^{(6)}(0) = 6!\binom{1/2}{2} = 6!\frac{1/2(1/2-1)}{2!} = -90$$

3. [2 points, 11.3 #55] Let $f(x, y, z) = xe^{3z} \cos 2y$. Determine the partial derivative:

$$\frac{\partial^3 f}{\partial x \partial y \partial z}.$$

$A.\, 6e^{2z} \cos 3y$ $\quad B.\, x^3 \cos 3y$ $\quad C.\, 5e^{2z} \cos 3y$

$D.\, 5e^{3z} \cos 2y$ $\quad$ **E.** $-\,6e^{3z} \sin 2y$ $\quad F.\, -\,xe^{3z} \sin 2y$

$$\frac{\partial f}{\partial z} = 3xe^{3z} \cos 2y$$
$$\frac{\partial^2 f}{\partial y \partial z} = -6xe^{3z} \sin 2y$$
$$\frac{\partial^3 f}{\partial x \partial y \partial z} = -6e^{3z} \sin 2y$$

4. [2 points, 8.7#23, 25] Find the Maclaurin series for the function:

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[Hint: First find the Maclaurin series for $g(t) = t^3 e^{2t}$.]

$A.\ \sum_{n=0}^{\infty} \frac{3^n}{n!(n+4)} x^{n+3}$ $B.\ \sum_{n=0}^{\infty} \frac{2^n}{n!(n+3)} x^{n+4}$ **C.** $\sum_{n=0}^{\infty} \frac{2^n}{n!(n+4)} x^{n+4}$

$D.\ \sum_{n=0}^{\infty} \frac{3^n}{n!(n+3)} x^{n+2}$ $E.\ \sum_{n=0}^{\infty} \frac{2^n}{n!(n+4)} x^{n+3}$ $F.\ \sum_{n=0}^{\infty} \frac{3^n}{n!(n+3)} x^{n+3}$

$$\begin{aligned}
e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
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&= \sum_{n=0}^{\infty} \frac{2^n}{n!} \frac{x^{n+4}}{n+4} = \sum_{n=0}^{\infty} \frac{2^n}{n!(n+4)} x^{n+4}
\end{aligned}$$

5. [2 points, 11.3 Lecture] Consider Laplace's equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Which of the following is a solution for it?

$A.\ z = \ln x \sin y \quad B.\ z = e^{x-y} \quad C.\ z = \sin x \cos y$

$\mathbf{D}.\ z = e^y \cos x \quad E.\ z = e^{x+y} \quad F.\ z = e^{x+y} \cos(x+y)$

The only correct solution is: $z = e^y \cos x$.

$$\frac{\partial z}{\partial x} = -e^y \sin x \qquad \frac{\partial z}{\partial y} = e^y \cos x$$

$$\frac{\partial^2 z}{\partial x^2} = -e^y \cos x \qquad \frac{\partial^2 z}{\partial y^2} = e^y \cos x$$

$$\implies \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

None of the other functions satisfies the equation.

6. [4 points, 11.1 #23] Consider the function:

$$f(x,y) = \frac{x^2}{9} + \frac{y^2}{4}$$

(a)[2 points]. Sketch the contour map of the function, using at least 3 level curves (with labels). We choose the level curves for $z = 1, z = 2, z = 3:$

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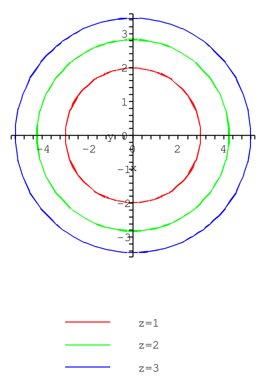
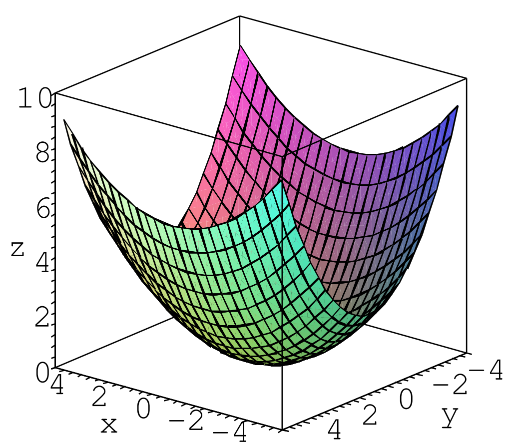


(b)[2 points]. Sketch the 3D-graph of the function.



7. [4 points, 11.3 Lecture] Consider the following function of two variables

$$f(x,y) = xye^{2y} + ye^{3xy}$$

(a)[2 points] Find the partial derivatives f_x and f_y at the point $(0,1)$.

$$f_x = ye^{2y} + 3y^2e^{3xy} \qquad f_x(0,1) = e^2 + 3$$

$$f_y = xe^{2y} + 2xye^{2y} + e^{3xy} + 3xye^{3xy} \qquad f_y(0,1) = 1$$

(b)[2 points] Find the second partial derivatives f_{xx}, f_{yy}, f_{xy}.

$$f_{xx} = 9y^3e^{3xy}$$

$$f_{xy} = e^{2y} + 2ye^{2y} + 6ye^{3xy} + 9xy^2e^{3xy}$$

$$f_{yy} = 4xe^{2y} + 4xye^{2y} + 6xe^{3xy} + 9x^2ye^{3xy}$$