

MAT 1322E, Winter 2005

MIDTERM TEST 2—Solutions

Family NAME:-----**Given NAME:**-----

Student Number:-----

MAX=18

- Time: 80 min. (Includes time for distribution of papers.)
- Only TI 30-type calculators are permitted. Notes or books are not permitted.
- There are 7 problems. Problems 1-5 are multiple choice. Circle the best possible answer. (Numerical answers are rounded to the given decimal.) Problems 6 and 7 require a detailed and clearly presented solutions.
- Work all problems in the space provided. Use the back-pages for rough work if necessary. Do not use any other paper. Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Work on multiple choice problems will be examined only in case of suspected fraud.

1. [2 points, 7.2 #19] Use Euler's method with step size 0.1 to estimate $y(0.2)$, where y is the solution of the initial value problem $y' = x^2 + 2y$, $y(0) = 1$.

A. 1.2 B. 1.441 C. 1.431 D. 1.686 E. 1.733 F. 1.676

Solution:

$$y(0.1) = 1 + 0.1(0^2 + 2(1)) = 1.2,$$

$$y(0.2) = 1.2 + 0.1(0.1^2 + 2(1.2)) = 1.441.$$

2. [2 points, 7.4 #1] A bacteria culture grows at a constant relative growth rate. After 1 hour there are 50 bacteria and after 2 hours the count is 100. Then the number of bacteria after 4 hours will be:

- A. 150 B. 200 C. 250 D. 300 E. 400 F. 800

Solution: We measure the time t in hours. Let $P(t)$ be the population at t hours, then we have

$$P(t) = P(0)e^{kt},$$

where k is the relative growth rate. Note that $P(1) = 50$ and $P(2) = 100$, we obtain

$$50 = P(0)e^k, \quad 100 = P(0)e^{2k}.$$

These imply that

$$P(0) = 25, \quad e^k = 2.$$

We thus have

$$P(t) = 25(2^t), \quad P(4) = 400.$$

3. [2 points, 8.2 #15] Determine whether the following series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n}$$

A. divergent B. 2 C. 3 D. 9 E. 27 F. 3.25

Solution: We have

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \sum_{n=1}^{\infty} 3 \cdot 3^n 4^{-n} = 3 \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

This series is a geometric series with first term $a = 9/4$ and common ratio $r = 3/4$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 3^{n+1}2^{-2n} = \frac{a}{1-r} = \frac{9/4}{1/4} = 9.$$

4. [2 points, 8.5 #6] Find the radius of the convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$$

A. 0 B. 1 C. 2 D. 3 E. 4 F. ∞

Solution: Let

$$a_n = \frac{(x-2)^n}{n}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot |(x-2)| = |x-2|.$$

Therefore $R = 1$.

5. [2 points, 7.5 #2] Consider the following logistic equation:

$$\frac{dP}{dt} = 20P \left(1 - \frac{P}{10} \right), \quad P(0) = 2.$$

The general solution will be:

A. $\frac{20}{1+4e^{-10t}}$ B. $\frac{10}{1+4e^{-20t}}$ C. $\frac{2}{1+0.25e^{-20t}}$ D. $\frac{10}{1+0.25e^{-20t}}$ E. $\frac{20}{1+4e^{10t}}$ F. $\frac{10}{1+4e^{20t}}$

Solution: Since $k = 20$, $K = 10$, this implies that $A = \frac{K-P(0)}{P(0)} = 4$. Therefore, the general solution is:

$$P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{10}{1 + 4e^{-20t}}.$$

6. [4 points, 7.3 #15] Suppose we have the equation $\frac{dy}{dt} = y(3t^2 + 2t + 1)$.

(a)[3 points]. Solve the equation (3 points).

Solution: This is a separable equation. We rewrite the equation as $\frac{dy}{y} = (3t^2 + 2t + 1) dt$. Integrating it gives

$$\ln |y| = t^3 + t^2 + t + C,$$

where C is a constant. Hence,

$$|y| = e^{t^3+t^2+t+C} = e^C e^{t^3+t^2+t},$$

$$y = \pm e^C e^{t^3+t^2+t} = A e^{t^3+t^2+t}.$$

(b)[1 points]. Find the solution satisfying the initial condition $y(1) = 1$.
From part a) we have

$$1 = A e^3,$$

which implies that $A = e^{-3}$, and so the special solution is

$$y = e^{-3} e^{t^3+t^2+t} = e^{t^3+t^2+t-3}.$$

7. [4 points; From lectures] Consider the following series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+2}{n^2+n+1}$$

(a)[2 points]. Test the series for convergence or divergence.

Solution: This is an alternating series. Let $b_n = \frac{n+2}{n^2+n+1}$. Then

1) $\lim_{n \rightarrow \infty} b_n = 0$;

2) Let $f(x) = \frac{x+2}{x^2+x+1}$ for $x \geq 1$, then $f(n) = b_n$ and

$$f'(x) = \frac{-x^2 - 4x - 1}{(x^2 + x + 1)^2} < 0.$$

Therefore $f(x)$ decreases for $x \geq 1$. In particular,

$$f(n) > f(n+1)$$

for all positive integer n . Hence b_n decreases.

By the Alternating Series Test, the series converges.

(b)[2 point]. Determine whether the series is absolutely convergent.

Solution:

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n+2}{n^2+n+1} \right| = \sum_{n=1}^{\infty} \frac{n+2}{n^2+n+1}.$$

Note that

$$\frac{n+2}{n^2+n+1} \geq \frac{n}{3n^2} = \frac{1}{3n}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{3n}$$

is divergent (the harmonic series), by Comparison Theorem, the original series is not absolutely convergent.

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1. [2 points, 7.2 #19] Use Euler's method with step size 0.1 to estimate $y(0.2)$, where y is the solution of the initial value problem $y' = x^2 + 3y$, $y(0) = 1$.

A. 1.3 B. 1.681 C. 1.691 D. 1.733 E. 2.022 F. 2.001

Solution:

$$y(0.1) = 1 + 0.1(0^2 + 3(1)) = 1.3,$$

$$y(0.2) = 1.3 + 0.1(0.1^2 + 3(1.3)) = 1.691.$$

2. [2 points, 7.4 #1] A bacteria culture grows at a constant relative growth rate. After 1 hour there are 50 bacteria and after 2 hours the count is 100. Then the number of bacteria after 5 hours will be:

- A. 150 B. 200 C. 250 D. 300 E. 400 F. 800

Solution: We measure the time t in hours. Let $P(t)$ be the population at t hours, then we have

$$P(t) = P(0)e^{kt},$$

where k is the relative growth rate. Note that $P(1) = 50$ and $P(2) = 100$, we obtain

$$50 = P(0)e^k, \quad 100 = P(0)e^{2k}.$$

These imply that

$$P(0) = 25, \quad e^k = 2.$$

We thus have

$$P(t) = 25(2^t), \quad P(5) = 800.$$

3. [2 points, 8.2 #15] Determine whether the following series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} 3^{n+2} 2^{-2n}$$

A. divergent B. 2 C. 3 D. 9 E. 27 F. 3.25

Solution: We have

$$\sum_{n=1}^{\infty} 3^{n+2} 2^{-2n} = \sum_{n=1}^{\infty} 9 \cdot 3^n 4^{-n} = 9 \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

This series is a geometric series with first term $a = 27/4$ and common ratio $r = 3/4$. Since $|r| < 1$, the series converges and we have

$$\sum_{n=1}^{\infty} 3^{n+1} 2^{-2n} = \frac{a}{1-r} = \frac{27/4}{1/4} = 27.$$

4. [2 points, 8.5 #6] Find the radius of the convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

A. ∞ B. 0 C. 1 D. 2 E. 3 F. 4

Solution: Let

$$a_n = \frac{(x-3)^n}{n}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot |(x-3)| = |x-3|.$$

Therefore $R = 1$.

5. [2 points, 7.5 #2] Consider the following logistic equation:

$$\frac{dP}{dt} = 30P \left(1 - \frac{P}{10} \right), \quad P(0) = 2.$$

The general solution will be:

A. $\frac{30}{1+4e^{-10t}}$ B. $\frac{10}{1+4e^{30t}}$ C. $\frac{3}{1+0.25e^{-30t}}$ D. $\frac{10}{1+0.25e^{-30t}}$ E. $\frac{30}{1+4e^{10t}}$ F. $\frac{10}{1+4e^{-30t}}$

Solution: Since $k = 30$, $K = 10$, this implies that $A = \frac{K-P(0)}{P(0)} = 4$. Therefore, the general solution is:

$$P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{10}{1 + 4e^{-30t}}.$$

6. [4 points, 7.3 #15] Suppose we have the equation $\frac{dy}{dt} = y(3t^2 + 2t + 2)$.

(a)[3 points]. Solve the equation.

Solution: This is a separable equation. We rewrite the equation as $\frac{dy}{y} = (3t^2 + 2t + 2) dt$. Integrating it gives

$$\ln |y| = t^3 + t^2 + 2t + C,$$

where C is a constant. Hence,

$$|y| = e^{t^3+t^2+2t+C} = e^C e^{t^3+t^2+2t},$$

$$y = \pm e^C e^{t^3+t^2+2t} = A e^{t^3+t^2+2t}.$$

(b)[1 points]. Find the solution satisfying the initial condition $y(1) = 1$.
From part a) we have

$$1 = A e^4,$$

which implies that $A = e^{-4}$, and so the special solution is

$$y = e^{-4} e^{t^3+t^2+2t} = e^{t^3+t^2+2t-4}.$$

7. [4 points; From lectures] Consider the following series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+3}{n^2+n+1}$$

(a)[2 points] Test the series for convergence or divergence.

Solution: This is an alternating series. Let $b_n = \frac{n+3}{n^2+n+1}$. Then

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Therefore $f(x)$ decreases for $x \geq 1$. In particular,

$$f(n) > f(n+1)$$

for all positive integer n . Hence b_n decreases.

By the Alternating Series Test, the series converges.

(b)[2 point]. Determine whether the series is absolutely convergent.

Solution:

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n+3}{n^2+n+1} \right| = \sum_{n=1}^{\infty} \frac{n+3}{n^2+n+1}.$$

Note that

$$\frac{n+3}{n^2+n+1} \geq \frac{n}{3n^2} = \frac{1}{3n}.$$

The series

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is divergent (the harmonic series), by Comparison Theorem, the original series is not absolutely convergent.