

Circle: DGD 1 (Andrew)

LAST NAME (in capitals):

DGD 2 (Elizabeth)

First name:

Marks: ~~12~~ 12

Student number:

MAT 1348A (Prof. M. Šajna) — Seventh Homework Assignment
Due March 18, 2015 by 4:00pm

Instructions: Print out this document and staple the pages. You may write on both sides of the paper or insert additional pages if necessary.

Submit a finished, presentable product. *Drafts and illegible papers will not be marked.* Show all relevant work to receive full credit.

Submit the assignment to your TA in the DGD or in the *submission box labeled MAT 1348A* in the Department of Mathematics and Statistics. Late assignments will not be accepted.

1. Let U be a non-empty universal set, and let \mathcal{R} be a binary relation on the set $\mathcal{P}(U)$ defined by

$$ARB \leftrightarrow A \cup B = U.$$

[4pts]

Determine whether \mathcal{R} is reflexive, symmetric, antisymmetric, or transitive. Fully justify each answer.

* \mathcal{R} is not reflexive:

Take any $A \subseteq U$ s.t. $A \neq U$ (counterexample)

Then $A \cup A = A \neq U$, so $A \not\mathcal{R} A$.

* \mathcal{R} is symmetric:

Take any $A, B \subseteq U$.

Assume $A \mathcal{R} B$. Then $A \cup B = U$.

By commutativity of unions of sets,

$B \cup A = A \cup B = U$, so $B \mathcal{R} A$.

* \mathcal{R} is not antisymmetric.

Counterexample: take any $A \subseteq U$, and

let $B = \bar{A}$. Then $A \cup B = A \cup \bar{A} = U$ and

$B \cup A = \bar{A} \cup A = U$, so $A \mathcal{R} B$ and $B \mathcal{R} A$.

However, $A \neq B = \bar{A}$ since $U \neq \emptyset$.

* R is not transitive:

Take any $A \subsetneq U$ and let $B = \bar{A}$, $C = A$.

Then $A \cup B = A \cup \bar{A} = U$ and $B \cup C = \bar{A} \cup A = U$.

Hence $A R B$ and $B R C$.

However, $A \cup C = A \cup A = A \neq U$, so $A \not R C$.

(counterexample).

2. A binary relation \mathcal{R} is defined on the set $\mathcal{F} = \{f : \mathbb{R} \rightarrow \mathbb{R}\}$ as follows:

[4pts]

$$(f, g) \in \mathcal{R} \iff f = cg \text{ for some constant } c \in \mathbb{Q}^+.$$

(Note: $f = cg$ means $f(x) = cg(x)$ for all $x \in \mathbb{R}$.) Prove that \mathcal{R} is an equivalence relation on \mathcal{F} .

* \mathcal{R} is reflexive:

take any $f \in \mathcal{F}$. Then $f = 1 \cdot f$ and $1 \in \mathbb{Q}^+$.

Hence $(f, f) \in \mathcal{R}$.

* \mathcal{R} is symmetric:

Take any $f, g \in \mathcal{F}$.

Assume $(f, g) \in \mathcal{R}$. Then there exists $c \in \mathbb{Q}^+$

s.t. $f = cg$. Since $c \neq 0$, we then have

$g = \frac{1}{c} f$, and $\frac{1}{c} \in \mathbb{Q}^+$. Hence $(g, f) \in \mathcal{R}$.

* \mathcal{R} is transitive:

Take any $f, g, h \in \mathcal{F}$.

Assume $(f, g), (g, h) \in \mathcal{R}$.

Then there exist $c, d \in \mathbb{Q}^+$ s.t. $f = cg$ and $g = dh$.

Hence $f = c(dh) = (cd)h$, and $cd \in \mathbb{Q}^+$.

We conclude $(f, h) \in \mathcal{R}$.

Since \mathcal{R} is reflexive, symmetric, and transitive,

it is an equivalence relation.

3. An equivalence relation \mathcal{R} is defined on the set \mathbb{R}^2 as follows:

[4pts]

$$(x, y)\mathcal{R}(x', y') \leftrightarrow 2x - y = 2x' - y'$$

Note: you need not prove that \mathcal{R} is an equivalence relation.

(a) Determine the partition of the set A below into equivalence classes of \mathcal{R} (restricted to A).

$$A = \left\{ \left(-\frac{1}{2}, -1\right), (0, -\pi), (0, 0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{\sqrt{2}}, 1\right), \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), (1, 1), (1, 2), (\pi, \pi) \right\}$$

$$S_1 = \left[\left(-\frac{1}{2}, -1\right) \right]_{\mathcal{R}} = \left\{ (x, y) \in A : 2x - y = 2\left(-\frac{1}{2}\right) - (-1) = 0 \right\} \\ = \left\{ \left(-\frac{1}{2}, -1\right), (0, 0), (1, 2) \right\}$$

$$S_2 = \left[(0, -\pi) \right]_{\mathcal{R}} = \left\{ (x, y) \in A : 2x - y = 2 \cdot 0 - (-\pi) = \pi \right\} \\ = \left\{ (0, -\pi), (\pi, \pi) \right\}$$

$$S_3 = \left[\left(\frac{1}{2}, 0\right) \right]_{\mathcal{R}} = \left\{ (x, y) \in A : 2x - y = 2 \cdot \frac{1}{2} - 0 = 1 \right\} \\ = \left\{ \left(\frac{1}{2}, 0\right), \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), (1, 1) \right\}$$

$$S_4 = \left[\left(\frac{1}{\sqrt{2}}, 1\right) \right]_{\mathcal{R}} = \left\{ \left(\frac{1}{\sqrt{2}}, 1\right) \right\} \quad \text{Partition: } \{S_1, S_2, S_3, S_4\}$$

(b) Give a geometric description of the equivalence class $[(1, -1)]_{\mathcal{R}}$ on \mathbb{R}^2 .

$$\left[(1, -1) \right]_{\mathcal{R}} = \left\{ (x, y) \in \mathbb{R}^2 : 2x - y = 2 \cdot 1 - (-1) = 3 \right\} \\ = \left\{ (x, y) \in \mathbb{R}^2 : 2x - y = 3 \right\} \\ = \left\{ (x, y) \in \mathbb{R}^2 : y = 2x - 3 \right\}$$

This is the straight line with equation $y = 2x - 3$.