

A

MAT 2322 D Winter 2016 February 25th, 14:30 Prof. Desjardins

TEST #1

Max = 20

Name: \_\_\_\_\_ Solutions

Student Number: \_\_\_\_\_

- Time: 80 min.
- Verify that your copy of the midterm has exactly 6 pages (including this one).
- There are 5 questions worth 4 marks each for a total of 20 points.
- Only basic scientific calculators are permitted: non-programmable, non-graphing.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
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Signature: \_\_\_\_\_

(A)

2

1. Find and classify the critical points of the following function.

$$f(x, y) = x^2 - xy^2 + 2y^2$$

$$f_x = 2x - y^2 \quad f_x = 0 \text{ if } y^2 = 2x$$

$$f_y = -2xy + 4y = 2y(2 - x)$$

$$f_y = 0 \text{ if } x = 2 \text{ or } y = 0$$

$$\text{if } x = 2, \quad y^2 = 4 \Rightarrow y = \pm 2 \Rightarrow (2, 2) \text{ and } (2, -2)$$

$$\text{if } y = 0, \quad x = 0 \Rightarrow (0, 0)$$

$$f_{xx} = 2, \quad f_{xy} = -2y, \quad f_{yy} = -2x + 4$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 2(-2x + 4) - (-2y)^2 \\ = 8 - 4x - 4y^2$$

$$D(0, 0) = 8 \quad f_{xx}(0, 0) = 2 \Rightarrow \boxed{(0, 0) \text{ local min}}$$

$$D(2, 2) = -16 \Rightarrow \boxed{(2, 2) \text{ saddle point}}$$

$$D(2, -2) = -16 \Rightarrow \boxed{(2, -2) \text{ saddle point}}$$

(A)

3

2. Use the method of Lagrange multipliers to determine the absolute minimum and maximum values of the function  $f(x, y) = x^2 - 4x + y^2$  on the circle  $x^2 + y^2 = 1$ .

$$f(x, y) = x^2 - 4x + y^2 \quad g(x, y) = x^2 + y^2 = 1$$

$$Df = \lambda Dg \Rightarrow f_x = \lambda g_x \Rightarrow 2x - 4 = 2\lambda x$$

$$\text{or } x - 2 = \lambda x \Rightarrow \lambda \neq 1$$

$$f_y = \lambda g_y \Rightarrow 2y = 2\lambda y$$

$$\text{or } y = \lambda y \Rightarrow y = 0$$

$$\text{if } y = 0, x^2 = 1 \Rightarrow x = \pm 1$$

so 2 points  $(1, 0)$  and  $(-1, 0)$

$$f(1, 0) = -3$$

$$f(-1, 0) = 5$$

$$\therefore \text{min } -3 \text{ and max } 5$$

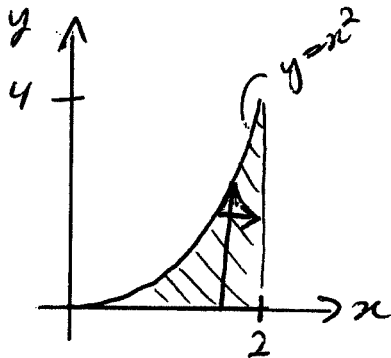
3. Calculate the following integral.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1} \, dx \, dy$$

have to switch order of integration

$x$  goes  $\sqrt{y}$  to 2, then  $y$  0 to 4

so region is



then  $y$  goes 0 to  $x^2$   
and  $x$  0 to 2

Thus

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1} \, dx \, dy$$

$$= \int_0^2 \int_0^{x^2} \sqrt{x^3+1} \, dy \, dx$$

$$= \int_0^2 \sqrt{x^3+1} (y \Big|_0^{x^2}) \, dx$$

$$= \int_0^2 x^2 \sqrt{x^3+1} \, dx$$

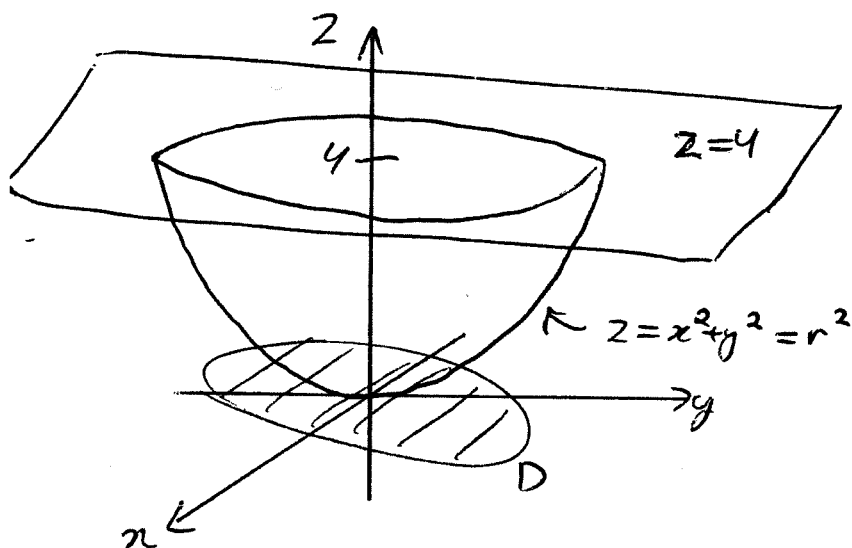
$$= \frac{2}{9} (x^3+1)^{3/2} \Big|_0^2$$

$$= \frac{2}{9} ((9)^{3/2} - 1) = \frac{2}{9} (27-1) = \boxed{\frac{52}{9}} \approx \boxed{5.78}$$

(A)

5

4. Compute the volume of the solid bounded by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$ .



shadow on  $xy$  plane  
is disk  $x^2 + y^2 \leq 4$

use polar coords

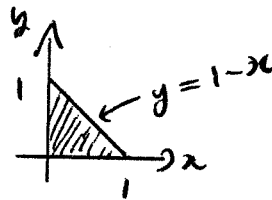
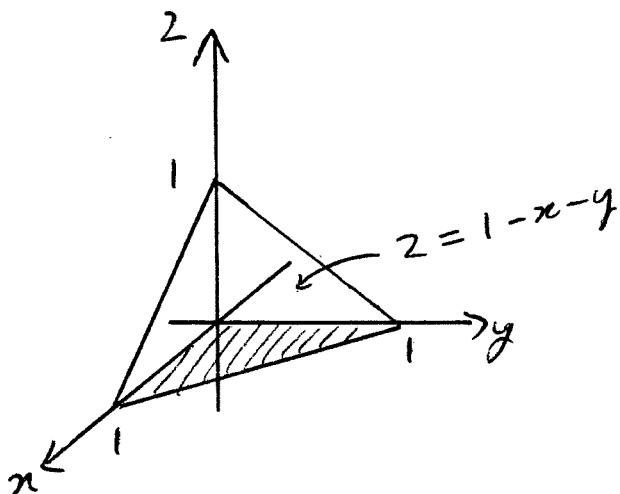
So Volume is

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta \\ &= 2\pi \int_0^2 (4r - r^3) \, dr \\ &= 2\pi \left( 2r^2 - \frac{1}{4}r^4 \Big|_0^2 \right) \\ &= 2\pi (8 - 4) \\ &= \boxed{8\pi} \approx \boxed{25.13} \end{aligned}$$

(A)

6

5. Calculate the total mass of the tetrahedron which is in the first octant and under the plane  $x + y + z = 1$  where the density function is  $\rho(x, y, z) = x$ .



The mass is  $M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx$

$$= \int_0^1 \int_0^{1-x} x \left( z \Big|_0^{1-x-y} \right) dy \, dx$$

$$= \int_0^1 \int_0^{1-x} (x - x^2 - xy) dy \, dx$$

$$= \int_0^1 \left( (x - x^2) - \frac{1}{2}xy^2 \Big|_0^{1-x} \right) dx$$

$$= \int_0^1 \left( (x - x^2)(1-x) - \frac{1}{2}x(1-x)^2 \right) dx$$

$$= \int_0^1 \frac{1}{2}x(1-x)^2 dx$$

$$= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) dx$$

$$= \frac{1}{2} \left( \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \Big|_0^1 \right)$$

$$= \frac{1}{2} \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{2} \left( \frac{6 - 8 + 3}{12} \right) = \boxed{\frac{1}{24}} \approx \boxed{0.042}$$

(B)

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(B)

2

1. Find and classify the critical points of the following function.

$$f(x, y) = 2x^2 - x^2y + y^2$$

$$f_x = 4x - 2xy = 2x(2-y)$$

$$f_x = 0 \text{ if } x=0 \text{ or } y=2$$

$$f_y = -x^2 + 2y \quad f_y = 0 \text{ if } 2y = x^2$$

$$\text{if } x=0, y=0 \Rightarrow (0,0)$$

$$\text{if } y=2, x^2=4 \Rightarrow x=\pm 2 \Rightarrow (2,2) \text{ and } (-2,2)$$

$$f_{xx} = 4 - 2y, \quad f_{xy} = -2x, \quad f_{yy} = 2$$

$$D = (4 - 2y)(2) - (-2x)^2 = 8 - 4y - 4x^2$$

$$D(0,0) = 8 \quad f_{xx}(0,0) = 4 \Rightarrow \boxed{(0,0) \text{ local min}}$$

$$D(2,2) = -16 \Rightarrow \boxed{(2,2) \text{ saddle point}}$$

$$D(-2,2) = -16 \Rightarrow \boxed{(-2,2) \text{ saddle point}}$$

(B)

3

2. Use the method of Lagrange multipliers to determine the absolute minimum and maximum values of the function  $f(x, y) = x^2 - 4y + y^2$  on the circle  $x^2 + y^2 = 4$ .

$$f(x, y) = x^2 - 4y + y^2, \quad g(x, y) = x^2 + y^2 = 4$$

$$\nabla f = \lambda \nabla g \Rightarrow f_x = \lambda g_x \Rightarrow 2x = 2\lambda x$$

$\text{or } x = \lambda x \Rightarrow x = 0$

$$f_y = \lambda g_y \Rightarrow -4 + 2y = 2\lambda y$$

$\text{or } -2 + y = \lambda y \Rightarrow \lambda \neq 1$

$$\text{if } x = 0, \quad y^2 = 4 \Rightarrow y = \pm 2$$

have 2 points  $(0, 2)$  and  $(0, -2)$

$$f(0, 2) = -4$$

$$f(0, -2) = 12$$

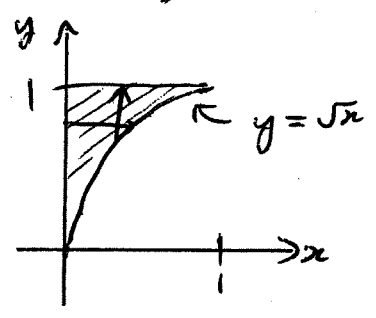
$\therefore$   $\boxed{\text{min } -4}$  and  $\boxed{\text{max } 12}$

3. Calculate the following integral.

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+3} dy dx$$

have to switch order of integration

y goes  $\sqrt{x}$  to 1, then x 0 to 1, so the region is



then x goes 0 to  $y^2$   
and y 0 to 1

thus  $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+3} dy dx$

$$= \int_0^1 \int_0^{y^2} \sqrt{y^3+3} dx dy$$

$$= \int_0^1 \sqrt{y^3+3} (x|_0^{y^2}) dy$$

$$= \int_0^1 y^2 \sqrt{y^3+3} dy$$

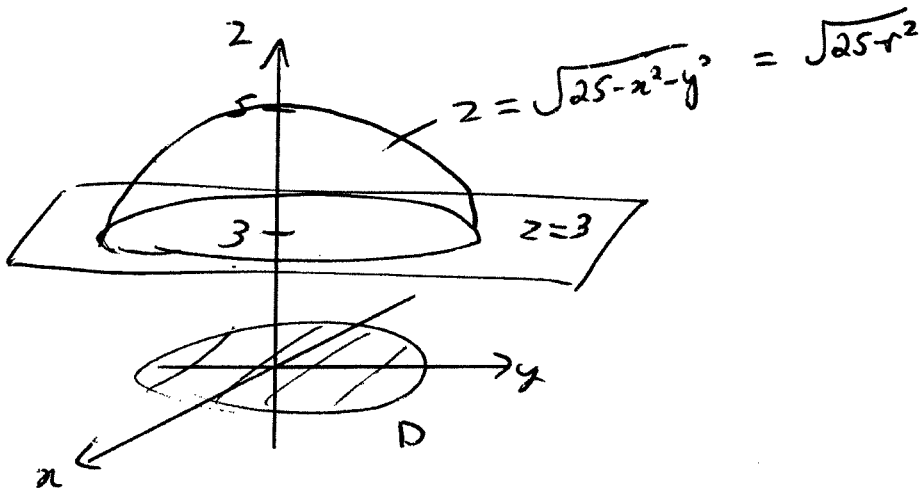
$$= \frac{2}{9} (y^3+3)^{3/2} \Big|_0^1$$

$$= \frac{2}{9} (4^{3/2} - 3^{3/2}) = \boxed{\frac{2}{9} (8 - 3\sqrt{3})} \approx \boxed{0.62}$$

(B)

5

4. Compute the volume of the solid bounded from above by the hemisphere  $z = \sqrt{25 - x^2 - y^2}$  and from below by the plane  $z = 3$ .



shadow on  $xy$  plane

is disk  $\sqrt{25 - (x^2 + y^2)} = 3$

or  $x^2 + y^2 \leq 16$

use polar coords

The volume is 
$$V = \int_0^{2\pi} \int_0^4 (\sqrt{25 - r^2} - 3) r dr d\theta$$

$$= 2\pi \int_0^4 (r\sqrt{25 - r^2} - 3r) dr$$

$$= 2\pi \left( -\frac{1}{3} (25 - r^2)^{3/2} - \frac{3}{2} r^2 \right) \Big|_0^4$$

$$= 2\pi \left( -\frac{1}{3} (9)^{3/2} - (25)^{3/2} - \frac{3}{2} (16) \right)$$

$$= 2\pi \left( -\frac{1}{3} (27 - 125) - 24 \right)$$

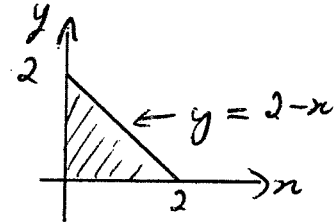
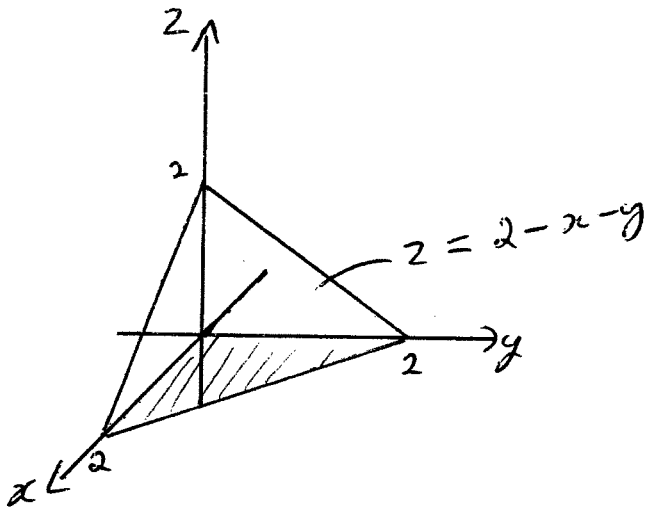
$$= 2\pi \left( \frac{98}{3} - 24 \right)$$

$$= \boxed{52\pi/3} \approx \boxed{54.45}$$

(B)

6

5. Calculate the total mass of the tetrahedron which is in the first octant and under the plane  $x + y + z = 2$  where the density function is  $\rho(x, y, z) = z$ .



the mass is  $M = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} z \, dz \, dy \, dx$

$$= \int_0^2 \int_0^{2-x} \left( \frac{1}{2} z^2 \Big|_0^{2-x-y} \right) dy \, dx$$

$$= \frac{1}{2} \int_0^2 \int_0^{2-x} (2-x-y)^2 dy \, dx$$

$$= \frac{1}{2} \int_0^2 \left( -\frac{1}{3} (2-x-y)^3 \Big|_0^{2-x} \right) dx$$

$$= -\frac{1}{6} \int_0^2 \left( (2-x - (2-x))^3 - (2-x)^3 \right) dx$$

$$= \frac{1}{6} \int_0^2 (2-x)^3 dx$$

$$= \frac{-1}{24} (2-x)^4 \Big|_0^2 = \frac{-1}{24} (0 - 16) = \boxed{\frac{2}{3}} \approx \boxed{0.67}$$