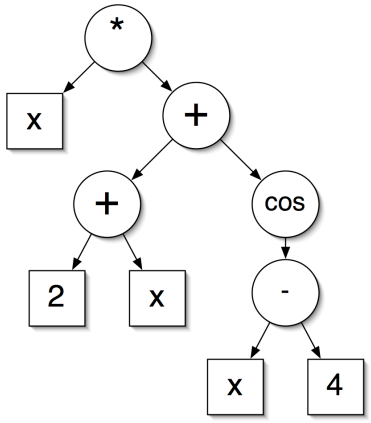
COMP 250 Fall 2015 - Homework #4

Due on November 11th at 23:59 (strict).

- Your solution must be returned electronically on MyCourse.
- The only format accepted for written answers is PDF. PDF files must open on SOCS computers. Any additional files (e.g. images must be included in the PDF). Do not submit a compressed repository including your files (e.g. rar or zip files). Upload instead each PDF or text file individually. Submissions that do not follow these guidelines will be penalized.
- The solution of programming questions must be written in java. Your program should compile and execute of SOCS computers. Java files that do not compile on SOCS computer will not be graded.
- To some extent, collaborations are allowed. These collaborations should not go as far as sharing code or giving away the answer. You must indicate on your assignments the names of the persons with who you collaborated or discussed your assignments (including members of the course staff). If you did not collaborate with anyone, you write "No collaborators" at the beginning of your document. Importantly, if asked, you should be able to orally explain your solution to a member of the course staff.
- Unless specified, all answers must be justified.
- When applicable, your pseudo-code should be commented and indented.
- The violation of these rules is subject to penalties.
- Partial answers will receive credits.
- We strongly recommend you to ask your questions as early as possible.
- Do not wait the last minute to submit your answers on MyCourse. Late submission of 24h or less will receive a penalty of 20%. This is a strict deadline. **Solutions not uploaded on MyCourse in time will not be graded.**

1) (40 points) Manipulation of symbolic expressions

In mathematics, a symbolic expression is an expression that contains variables like x, y... For example, 4 * x + 3 * y and sin(2 * x) + e^x are symbolic expressions. We have seen in class how such expressions can be represented using a (improper) binary tree, where internal nodes correspond to operator (+, -, *, cos, sin, exp) and the children of a node are the operands. For example, the expression x * ((2 + x) + cos(x-4)) can be represented as:



The code available at http://www.cs.mcgill.ca/~blanchem/250/hw4 provides you with a binary tree data structure for storing expressions. It contains a constructor that takes as input a string describing an expression and returns the root of the binary tree storing that expression. The string describing the expression is in the following format.
(1) The only operators considered are add, mult, minus, sin, cos, and exp.
(2) Each operator is followed by its operand(s), in parentheses. If there are two operands, they are separated by a comma.
(3) We only consider expressions with a single variable called x.
Thus, the mathematical expression x * ((2 + x) + cos(x - 4)) should be written as: mult(x,add(add(2,x),cos(minus(x,4)))). So,
new treeNode("mult(x,add(add(2,x),cos(minus(x,4)))"); returns the root of the tree above.
Another example of notation: the expression cos(3.1416*x + e^(sin (x – 1))) should be written as: cos(add(mult(3.1416,x),exp(sin(minus(x,1)))))

You are also provided with a method deepcopy() which returns a copy of the subtree rooted at the node on which it is called. More precisely, it builds a completely new tree, with new nodes, that is a copy of the original. Make sure you understand what it does, as it is going to be very useful. Finally, the treeNode class is equipped with a toString() method which prints the whole subtree rooted at a node. This will be useful for debugging.

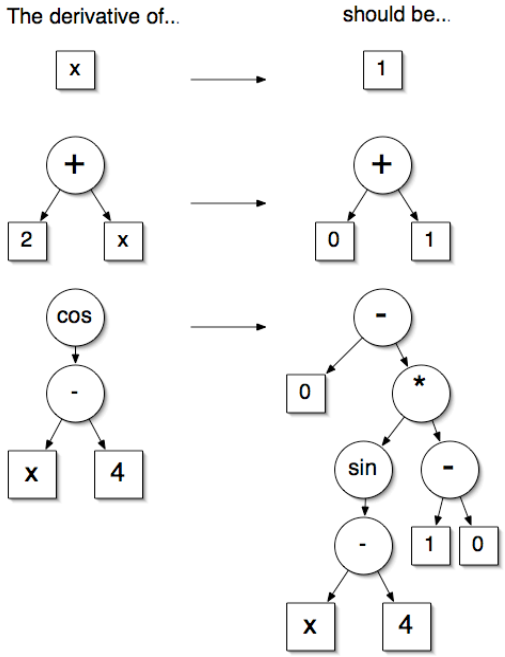
Two useful things in Java:

1) If you want to test whether two objects of type String are equal, use the equals method. For example:
   String word1 = "hello";
   String word2 = "goodbye";
   if (word1.equals(word2)) { …}
   if (word1.equals("hello")){…}

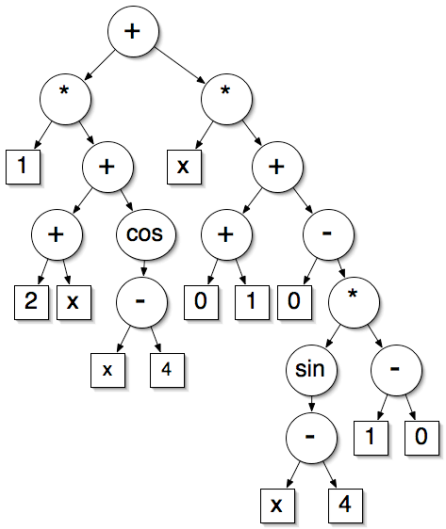
2) All nodes in our expression tree are stored as objects of type String. If an object of type String actually describes a number, it can be turned into a double as follows:
   String myNumber = "3.1416";
   double x=Double.parseDouble(myNumber);

a) (15 points) Complete the evaluate(double x) method, which returns the value of the expression rooted at the treeNode on which it called, for a particular value of x. For example, evaluate(1) on the tree above should return approximately 2.01.... For sin and cos, angles are measured in radians.

b) (25 points) Complete the differentiate() method, which returns a new expression tree describing the derivative of the expression described by the tree rooted at the treeNode on which it is called. Notice that the original tree should remain intact and that the tree containing the derivative should share no node with the original. For example:



The derivative of the original tree is thus:



Notice that the tree above may seem more complicated than necessary. However, it is the one you will most likely end up with after differentiating the original expression. (It could be slightly different but equivalent, and that would be OK). Do not worry about simplifying the expression described by the tree.

Again, that sounds really hard, but it's not, because the rules of derivation lend themselves very well to the tree representation we are using. Indeed, the chain rule for derivation is a recursive algorithm!

d/dx (f(x) + g(x)) = d/dx f(x) + d/dx g(x)
d/dx (f(x) – g(x)) = d/dx f(x) – d/dx g(x)
d/dx (f(x) * g(x)) = (d/dx f(x)) * g(x) + f(x) * (d/dx g(x))
d/dx (sin(f(x))) = cos(f(x)) * d/dx f(x)
d/dx (cos(f(x))) = - sin (f(x)) * d/dx f(x)
d/dx (exp(f(x))) = exp(f(x)) * d/dx f(x)

with the base cases:

d/dx (x) = 1
d/dx (c) = 0 for any constant c

2) (10 points) Binary search trees

Consider a binary search tree that contains *n* nodes with keys 1, 2, 3, ... , *n*.
The shape of the tree depends on the order in which the keys have been inserted in the tree.

a) In what order should the keys be inserted into the binary search tree to obtain a tree with minimal height?

b) On the tree obtained in (a), what would be the worst-case running time of a find, insert, or remove operation? Use the big-Oh notation.

c) In what order should the keys be inserted into the binary search tree to obtain a tree with maximal height?

d) On the tree obtained in (c), what would be the worst-case running time of a find, insert, or remove operation? Use the big-Oh notation.

3) (15 points) Finding the k-th element

Consider the following problem: Given an **unsorted** array A[0...n-1] of distinct integers, find the k-th smallest element (with k=0 giving the smallest element).
For example, findKth([9 10 6 7 4 12] , 0) = 4 and findKth([1 7 6 4 15 10] , 4) = 10.

a) (10 points). Complete the following pseudocode for the findKth algorithm. Your algorithm should have similarities to quickSort and to binarySearch and should call the partition algorithm described in class and used by the quickSort method. You can call the

partition without redefining it. Your algorithm should have the running time described in (b).

Algorithm findKth(A, start, stop, k)
Input: An unsorted array A[start...stop] of numbers, and a number k between 0 and stop-start-1
Output: Returns the k-th smallest element of A[start...stop]
/* Complete this pseudocode */

b) (5 points) Assuming that n is a power of two and that we are always lucky enough that the pivot chosen by the partition method always splits A[start...stop] into two equal halves, show that your algorithm runs in time O(n). You can assume that executing partition(start,stop) takes O(stop-start+1) time.

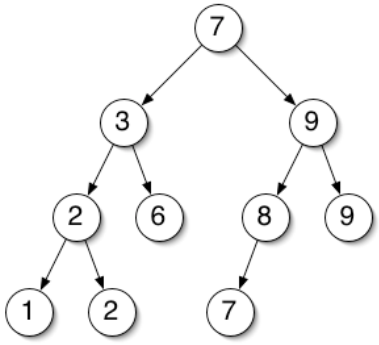
4) **(20 points)** Tree traversals

Consider the following pair of recursive algorithms calling each other to traverse a binary tree.

```
Algorithm weirdPreOrder(treeNode n)
    if (n != null) then
        print n.getValue()
        weirdPostOrder( n.getRightChild() )
        weirdPreOrder( n.getLeftChild() )
```

```
Algorithm weirdPostOrder(treeNode n)
    if (n != null) then
        weirdPreOrder( n.getRightChild() )
        weirdPostOrder( n.getLeftChild() )
        print n.getValue()
```

a) **(5 points)** Write the output being printed when weirdPreOrder(root) is executed on the following binary tree:



b) **(5 points)** Write the output being printed when weirdPostOrder(root) is executed.

c) **(5 points)** Consider the binary tree traversal algorithm below.
Algorithm queueTraversal(treeNode n)
Input: a treeNode n

Output: Prints the value of each node in the binary tree rooted at n

```
Queue q ← new Queue();
q.enqueue(n);
while (! q.empty() ) do
    x ← q.dequeue();
    print x.getValue();
    if ( x.getLeftChild() != null ) then q.enqueue( x.getLeftChild() );
    if ( x.getRightChild() != null ) then q.enqueue( x.getRightChild() );
```

Question: Write the output being printed when queueTraversal(root) is executed.

d) **(5 points)** Consider the binary tree traversal algorithm below.

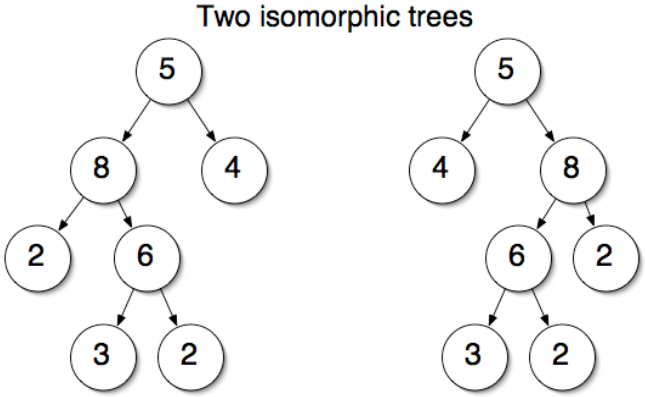
Algorithm stackTraversal(treeNode n)
Input: a treeNode n
Output: Prints the value of each node in the binary tree rooted at n

```
Stack s ← new Stack();
s.push(n);
while (! s.empty() ) do
    x ← s.pop();
    print x.getValue();
    if (x.getLeftChild() != null) then s.push(x.getLeftChild());
    if (x.getRightChild() != null) then s.push(x.getRightChild());
```

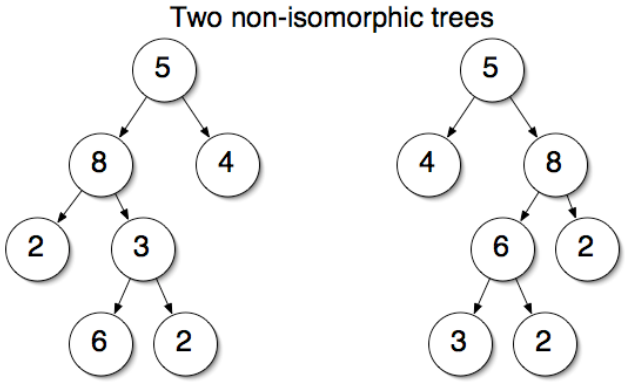
Question: Write the output being printed when stackTraversal(root) is executed. This is similar to what traversal method seen previously in class?

5) **(15 points)** Tree isomorphism

Two unordered binary trees A and B are said to be *isomorphic* if, by swapping the left and right subtrees of certain nodes of A, one can obtain a tree identical to B. For example, the following two trees are isomorphic:



because starting from the first tree and exchanging the left and right subtrees of node 5 and of node 8, one obtains the second tree. On the other hand, the following two trees are not isomorphic, because it is impossible to rearrange one into the other:



Question: Write a recursive algorithm that tests if the trees rooted at two given treeNodes are isomorphic. Hint: if your algorithm takes more than 10 lines to write, you're probably not doing the right thing.

Algorithm isIsomorphic(treeNode A, treeNode B)
Input: Two treeNodes A and B
Output: Returns true if the trees rooted at A and B are isomorphic
/* Complete this pseudocode */