RYERSON UNIVERSITY
DEPARTMENT
OF
MATHEMATICS

MTH 210 **Midterm Test (Version 1)** **March 5, 2012**

Total marks: 40 **Time allowed: 50 Minutes**

NAME (Print): SOLUTIONS **Student #:** ____________

Circle your lab section:

1	2	3	4
Thur. 12:00-13:00	Thur., 13:00-14:00	Wed. 10:00-11:00	Mon. 14:00-15:00
KHE 323	ENG LG13	TRS 2164	KHE 129

Instructions:

- Verify that your paper contains 5 questions on 6 page (including this page).
- Electronic devices such as calculators, cellphones and iPods must be turned off and kept inaccessible during the test.
- Please keep your Ryerson photo ID card displayed on your desk during the test.
- In every question show all your work. The correct answer alone may be worth nothing.
- Delete all irrelevant and incorrect work because marks may be deducted for work which is misleading, irrelevant or incorrect, even if steps for a correct solution are also shown.
- Please write only in this booklet. Use of scrap paper or additional enclosures is not allowed. If you need more space continue on the back of the page, directing marker where the answer continues with a bold sign.
- You may assume the following formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

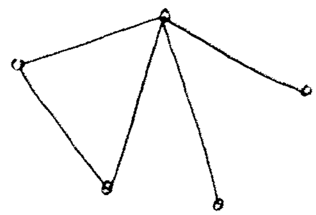
$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

- Recall the definition of divides:

$$a \mid b \Leftrightarrow \exists m \in \mathbb{Z} \text{ such that } b = am.$$

1. **(12 points)** For each of the following, either draw a graph with the specified properties, or else explain why no such graph exists.

   (a) **(3 points)** Simple graph with five vertices of degrees 1, 1, 2, 2, 4.

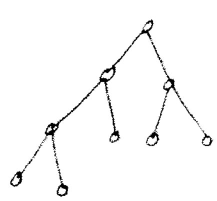
   (b) **(3 points)** Graph with six vertices of degrees 1, 1, 2, 2, 4, 5.

   Does not exist: cannot have total degree odd.

   (c) **(3 points)** Tree with six vertices and six edges.

   Does not exist: a tree with six vertices has only five edges.

   (d) **(3 points)** Full binary tree with four internal vertices and five leaves.

2. **(9 points)** Use mathematical induction to prove that $9^n - 2^n$ is divisible by 7 for all integers $n \geq 0$.

For integers n, let $P(n)$ denote the predicate "$9^n - 2^n$ is divisible by 7".

Basis step. Since $9^0 - 2^0 = 1 - 1 = 0 = 7 \cdot 0$, $P(0)$ is true.

Inductive step.

Induction hypothesis: Suppose that $k \geq 0$ is an integer such that $P(k)$ is true, i.e. $7 \mid (9^k - 2^k)$.

We must show that $P(k+1)$ is true, i.e. that $7 \mid (9^{k+1} - 2^{k+1})$.

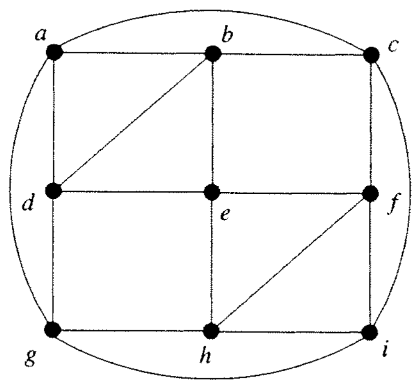
Note that, by the induction hypothesis, $9^k - 2^k = 7t$ for some $t \in \mathbb{Z}$. Now,

$$
\begin{aligned}
9^{k+1} - 2^{k+1} &= 9 \cdot 9^k - 2 \cdot 2^k \\
&= 7 \cdot 9^k + 2 \cdot 9^k - 2 \cdot 2^k \\
&= 7 \cdot 9^k + 2(9^k - 2^k) \\
&= 7 \cdot 9^k + 2(7t) \\
&= 7(9^k + 2t).
\end{aligned}
$$

Since $9^k + 2t$ is an integer, it follows that $7 \mid (9^{k+1} - 2^{k+1})$, i.e. $P(k+1)$ is true.

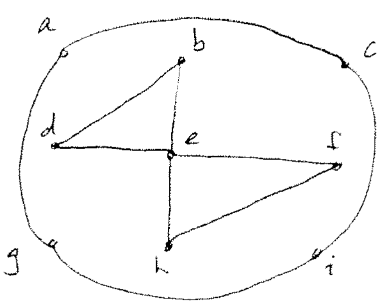
By the Principle of Mathematical Induction, $P(n)$ is true for all integers $n \geq 0$.

3. **(6 points)** Does the following graph have an Euler circuit? If so, find one. If not, explain why not.



Yes, since the graph is connected and each vertex has even degree.

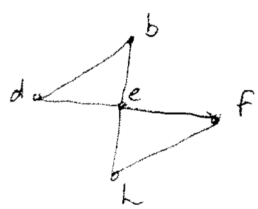
Let C = abcfihgda. Removing the edges of C:



Let C' = aciga. Form C'' by pasting C and C' together:

C'' = acigabcfihgda

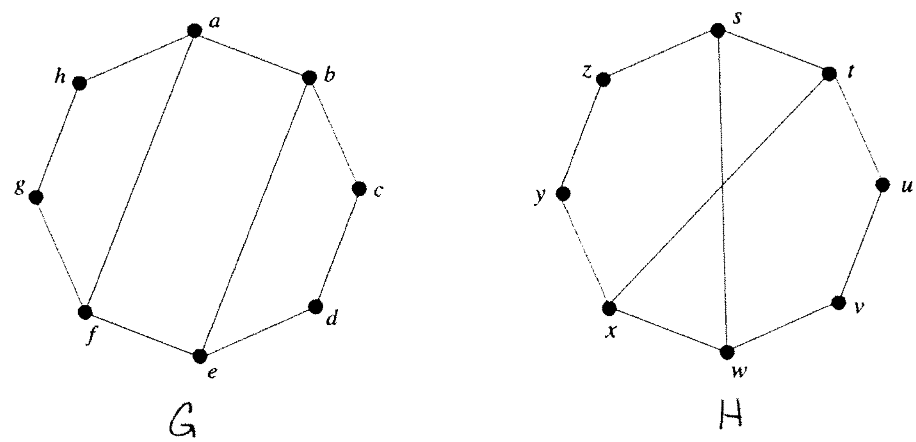
Take C = C''. The edges remaining:



Let C' = befhedb. Pasting together C + C' gives

$\boxed{\text{C'' = acigabefhedbcfihgda.}}$ ← Euler circuit.

4. **(5 points)** Determine whether the following graphs are isomorphic. If so, give an isomorphism. If not, explain why not.



The graphs are not isomorphic, since H contains a simple circuit of length 5 (eg. tuvwxt), but G does not.

5. **(8 points)** Consider the sequence defined recursively by

$$a_k = 3 + 2a_{k-1} \text{ for all integers } k \geq 2,\ a_1 = 1$$

(a) **(3 points)** Find a_2, a_3, a_4. Keep your intermediate answers as you will need them for the next part of this question.

$$a_2 = 3 + 2 \cdot 1 = 3 + 2 = 5$$

$$a_3 = 3 + 2(3+2) = 3 + 2 \cdot 3 + 2^2 = 13$$

$$a_4 = 3 + 2(3 + 2 \cdot 3 + 2^2) = 3 + 2 \cdot 3 + 2^2 \cdot 3 + 2^3 = 29$$

(b) **(5 points)** Use iteration to find a general formula for a_n. (You may work out additional terms if you wish.) Simplify your answer as much as possible, showing your work. Your final answer should not contain a summation symbol or an ellipsis $(\cdots)$.

$$a_n = 3 + 2 \cdot 3 + 2^2 \cdot 3 + \cdots + 2^{n-2} \cdot 3 + 2^{n-1}$$

$$= 3(1 + 2 + 2^2 + \cdots + 2^{n-2}) + 2^{n-1}$$

$$= 3\left(\frac{2^{n-1} - 1}{2 - 1}\right) + 2^{n-1}$$

$$= 3 \cdot 2^{n-1} - 3 + 2^{n-1}$$

$$= 4 \cdot 2^{n-1} - 3$$

$$= 2^2 \cdot 2^{n-1} - 3$$

$$= 2^{n+1} - 3$$