Midterm solutions
MATH 209 Fall 2014

1.(a)

$$\lim_{x\to 2}(-x) = -2 \tag{1}$$
$$\lim_{x\to 2}(x-2)^2 = 0\ . \tag{2}$$

Therefore a numerical limit does not exist, the absolute value of the functional value of the ratio goes to ∞:

$$\lim_{x\to 2}\left|\frac{-x}{(x-2)^2}\right| = \infty\ . \tag{3}$$

Since

$$\frac{-x}{(x-2)^2} < 0 \tag{4}$$

for x in $(0,2)\cup(2,\infty)$ we have

$$\lim_{x\to 2^-}\frac{-x}{(x-2)^2} = -\infty \tag{5}$$
$$\lim_{x\to 2^+}\frac{-x}{(x-2)^2} = -\infty \tag{6}$$

and therefore we have

$$\boxed{\lim_{x\to 2}\frac{-x}{(x-2)^2} = -\infty\ .} \tag{7}$$

But remember that, strictly speaking, the limit does not exist, the above formula only expresses that the functional values get arbitrarily large and negative as $x \to 2$ from either side.

1.(b) Since

$$\lim_{x\to 5}\left(\sqrt{x-1}-2\right) = \sqrt{5-1}-2 = \sqrt{4}-2 = 0 \tag{8}$$
$$\lim_{x\to 5}(x-5) = 0\ , \tag{9}$$

the limit of their ratio is of 0/0 indeterminate form.

$$\frac{\sqrt{x-1}-2}{x-5} = \frac{\sqrt{x-1}-2}{x-5}\cdot\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2} \tag{10}$$
$$= \frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{(x-5)(\sqrt{x-1}+2)} \tag{11}$$
$$= \frac{(\sqrt{x-1})^2-2^2}{(x-5)(\sqrt{x-1}+2)} \tag{12}$$
$$= \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)} \tag{13}$$
$$= \frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x-1}+2)} \tag{14}$$
$$= \frac{1}{\sqrt{x-1}+2}\ . \tag{15}$$

$$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5} = \lim_{x\to 5}\frac{1}{\sqrt{x-1}+2} \tag{16}$$
$$= \frac{1}{\sqrt{5-1}+2} \tag{17}$$
$$= \frac{1}{\sqrt{4}+2} \tag{18}$$
$$= \frac{1}{2+2} \tag{19}$$
$$= \boxed{\frac{1}{4}}\ . \tag{20}$$

1.(c)

$$\lim_{x\to\infty}\frac{3x^2-2x-1}{4x^3-5x} = \lim_{x\to\infty}\frac{3x^2\left(1-\frac{2}{3x}-\frac{1}{3x^2}\right)}{4x^3\left(1-\frac{5}{4x^2}\right)} \tag{21}$$
$$= \lim_{x\to\infty}\frac{3x^2}{4x^3}\cdot\frac{\left(1-\frac{2}{3x}-\frac{1}{3x^2}\right)}{\left(1-\frac{5}{4x^2}\right)} \tag{22}$$
$$= \lim_{x\to\infty}\frac{3}{4x}\cdot\lim_{x\to\infty}\frac{\left(1-\frac{2}{3x}-\frac{1}{3x^2}\right)}{\left(1-\frac{5}{4x^2}\right)} \tag{23}$$
$$= 0\cdot 1 \tag{24}$$
$$= \boxed{0}\ . \tag{25}$$

2.

$$\lim_{t\to 0}\frac{g(x+t)-g(x)}{t} \tag{26}$$
$$=\lim_{t\to 0}\frac{\left[4(x+t)-(x+t)^2\right]-\left[4x-x^2\right]}{t} \tag{27}$$
$$=\lim_{t\to 0}\frac{\left[4(x+t)-(x^2+2xt+t^2)\right]-\left[4x-x^2\right]}{t} \tag{28}$$
$$=\lim_{t\to 0}\frac{4x+4t-x^2-2xt-t^2-4x+x^2}{t} \tag{29}$$
$$=\lim_{t\to 0}\frac{\cancel{4x}+4t-\cancel{x^2}-2xt-t^2-\cancel{4x}+\cancel{x^2}}{t} \tag{30}$$
$$=\lim_{t\to 0}\frac{4t-2xt-t^2}{t} \tag{31}$$
$$=\lim_{t\to 0}\frac{\cancel{t}(4-2x-t)}{\cancel{t}} \tag{32}$$
$$=\lim_{t\to 0}(4-2x-t) \tag{33}$$
$$=4-2x\ . \tag{34}$$

Therefore

$$\boxed{g'(x)=4-2x}\ . \tag{35}$$

3.(a)

$$f'(x)=\left(7\sqrt[6]{x^3}-\frac{1}{x^5}\right)' \tag{36}$$
$$=\left(7x^{\frac{3}{6}}-x^{-5}\right)' \tag{37}$$
$$=\left(7x^{\frac{1}{2}}-x^{-5}\right)' \tag{38}$$
$$=7\cdot\frac{1}{2}x^{\frac{1}{2}-1}-(-5)x^{-5-1} \tag{39}$$
$$=\frac{7}{2}x^{-\frac{1}{2}}+5x^{-6}\ . \tag{40}$$

$$f'(1)=\frac{7}{2}\cdot 1^{-\frac{1}{2}}+5\cdot 1^{-6} \tag{41}$$
$$=\frac{7}{2}+5 \tag{42}$$
$$=\boxed{\frac{17}{2}}\ . \tag{43}$$

3.(b)

$$g'(x) \tag{44}$$
$$=[1+3\ln(x^2)]'[3x^4-4]+[1+3\ln(x^2)][3x^4-4]' \tag{45}$$
$$=3(\ln(x^2))'\cdot[3x^4-4]+[1+3\ln(x^2)]\cdot 12x^3 \tag{46}$$
$$=3\frac{1}{x^2}(x^2)'\cdot[3x^4-4]+[1+3\ln(x^2)]\cdot 12x^3 \tag{47}$$
$$=3\frac{1}{x^2}\cdot 2x\cdot[3x^4-4]+[1+3\ln(x^2)]\cdot 12x^3 \tag{48}$$
$$=\frac{6}{x}\cdot[3x^4-4]+[1+3\ln(x^2)]\cdot 12x^3\ . \tag{49}$$

$$g'(2)=\frac{6}{2}\cdot[3\cdot 2^4-4]+[1+3\ln(2^2)]\cdot 12\cdot(2)^3 \tag{50}$$
$$=3\cdot 44+[1+3\ln(2^2)]\cdot 96 \tag{51}$$
$$=\boxed{132+[1+3\ln(2^2)]\cdot 96}\ . \tag{52}$$

3.(c)

$$h'(x)=\left(\frac{x^3-3}{3x+5}\right)' \tag{53}$$
$$=\frac{(x^3-3)'(3x+5)-(x^3-3)(3x+5)'}{(3x+5)^2} \tag{54}$$
$$=\boxed{\frac{3x^2\cdot(3x+5)-(x^3-3)\cdot 3}{(3x+5)^2}}\ . \tag{55}$$

3.(d)

$$y(x)=\sqrt{2x+8}\ . \tag{56}$$

$$y'(x)=\left(\sqrt{2x+8}\right)' \tag{57}$$
$$=\left((2x+8)^{\frac{1}{2}}\right)' \tag{58}$$
$$=\frac{1}{2}(2x+8)^{\frac{1}{2}-1}(2x+8)' \tag{59}$$
$$=(2x+8)^{-\frac{1}{2}}\ . \tag{60}$$

$$dy=y'(x)dx \tag{61}$$
$$=(2x+8)^{-\frac{1}{2}}dx \tag{62}$$
$$\tag{63}$$

With

$$x = 4, \qquad dx = 0.1 \ , \tag{64}$$

we have

$$\begin{aligned} dy &= (2 \cdot 4 + 8)^{-\frac{1}{2}} \cdot 0.1 &(65)\\ &= (16)^{-\frac{1}{2}} \cdot 0.1 &(66)\\ &= \frac{1}{4} \cdot 0.1 &(67)\\ &= \boxed{0.025} \ . &(68) \end{aligned}$$

4.

$$\begin{aligned} C'(x) &= (5000 + 2x)' = 2 &(69)\\ R'(x) &= \left(10x - \frac{x^2}{1000}\right)' &(70)\\ &= 10 - \frac{2x}{1000} &(71)\\ &= 10 - \frac{x}{500} \ . &(72) \end{aligned}$$

$$\begin{aligned} P(x) &= R(x) - C(x) &(73)\\ &= 8x - \frac{x^2}{1000} - 5000 &(74) \end{aligned}$$

$$\begin{aligned} P'(x) &= R'(x) - C'(x) &(75)\\ &= 8 - \frac{x}{500} \ . &(76) \end{aligned}$$

$$\begin{aligned} R(2010) &\approx R(2000) + R'(2000) \cdot (2010 - 2000) &(77)\\ &= \left(10 \cdot 2000 - \frac{2000^2}{1000}\right) + \left(10 - \frac{2000}{500}\right) \cdot 10 &(78)\\ &= 16000 + 60 &(79)\\ &= \boxed{16060} \ . &(80) \end{aligned}$$

$$\begin{aligned} P(2010) &\approx P(2000) + P'(2000) \cdot (2010 - 2000) &(81)\\ &= \left(8 \cdot 2000 - \frac{2000^2}{1000} - 5000\right) + \left(8 - \frac{2000}{500}\right) \cdot 10 &(82)\\ &= 7000 + 40 &(83)\\ &= \boxed{7040} \ . &(84) \end{aligned}$$

5.(a) The average cost function:

$$\begin{aligned} \overline{C}(x) &= \frac{C(x)}{x} &(85)\\ &= \frac{10000 + 200x - \frac{1}{10}x^2}{x} &(86)\\ &= \boxed{\frac{10000}{x} + 200 - \frac{1}{10}x} \ . &(87) \end{aligned}$$

$$\begin{aligned} \overline{C}(200) &= \frac{10000}{200} + 200 - \frac{1}{10} \cdot 200 &(88)\\ &= 50 + 200 - 20 &(89)\\ &= \boxed{230} \ . &(90) \end{aligned}$$

5.(b) The exact average cost of producing the 201st washing machine:

$$\begin{aligned} \overline{C}(201) - \overline{C}(200) &= \left(\frac{10000}{201} + 200 - \frac{1}{10} \cdot 201\right) &(91)\\ &\quad - \left(\frac{10000}{200} + 200 - \frac{1}{10} \cdot 200\right) &(92)\\ &= \boxed{-0.349} \ . &(93) \end{aligned}$$

5.(c) The marginal average cost function:

$$\begin{aligned} \overline{C}'(x) &= \left(\frac{10000}{x} + 200 - \frac{1}{10}x\right)' &(94)\\ &= -\frac{10000}{x^2} - \frac{1}{10} &(95) \end{aligned}$$

$$\begin{aligned} \overline{C}(201) - \overline{C}(200) &\approx \overline{C}'(200) \cdot (201 - 200) &(96)\\ &\left(-\frac{10000}{200^2} - \frac{1}{10}\right) \cdot 1 &(97)\\ &= \boxed{-0.35} \ . &(98) \end{aligned}$$

6.

$$x(t)^2 - t \cdot x(t)^2 + t^3 + 11 = 0 \tag{99}$$
$$2x(t)x'(t) - x(t)^2 - t \cdot 2x(t)x'(t) + 3t^2 = 0 \tag{100}$$
$$2x(t)x'(t) - t \cdot 2x(t)x'(t) = x(t)^2 - 3t^2 \tag{101}$$
$$2x(t)x'(t)(1-t) = x(t)^2 - 3t^2 \tag{102}$$
$$x'(t) = \frac{x(t)^2 - 3t^2}{2x(t)(1-t)} \tag{103}$$
$$x' = \frac{x^2 - 3t^2}{2x(1-t)}\ . \tag{104}$$

At $(t, x) = (-2, 1)$ we have

$$x' = \frac{x^2 - 3t^2}{2x(1-t)} \tag{105}$$
$$= \frac{1^2 - 3 \cdot (-2)^2}{2 \cdot 1 \cdot (1 - (-2))} \tag{106}$$
$$= \frac{1 - 12}{6} \tag{107}$$
$$= \boxed{-\frac{11}{6}}\ . \tag{108}$$

---

7. Independent variable: t (time), dependent variables: $r(t)$ (radius), $A(t)$ (surface area)

$$A(t) = 4\pi r(t)^2 \tag{109}$$
$$A'(t) = 8\pi r(t) r'(t) \tag{110}$$

For

$$r(t) = 10, \quad r'(t) = 3 \tag{111}$$

we have

$$A(t) = 8\pi \cdot 10 \cdot 3 = 240\pi = \boxed{753.98 \frac{\text{cm}^2}{\text{min}}}\ . \tag{112}$$