Microeconomics Study Guide

Prepared for ECON 4020 A
Professor K. G. Armstrong
Department of Economics
Carleton University

Compiled by Briana Brownell

Table of Contents

Mathematical Preliminaries

1. Graph the following family of functions:

    a. $y = e^x$

    b. $y = -e^x$

    c. $y = e^{-x}$

    d. $y = -e^{-x}$

2. Graph the following family of functions:

    a. $y = e^x + 1$

    b. $y = e^{x+1}$

    c. $y = e^{-x+1}$

    d. $y = e^{-x} - 1$

3. Graph the following family of functions:

    a. $y = \frac{e^x}{2}$

    b. $y = e^{x/2}$

    c. $y = 2e^x$

    d. $y = e^{2x}$

4. Graph the following:
    a. $y < e^x + a \text{ where } a > 0$
    b. $y > ax^2 \text{where } a > 0$
    c. $y \leq (x - a)^2 - b \text{ where } a > 0; \mathrm{b} > 0$

5. Graph the following inequalities:
    a. $y > x + 1 \;\wedge\; \mathrm{x} \geq -1$
    b. $x + y > 1 \;\wedge\; \mathrm{x} - \mathrm{y} > 0$
    c. $2x - y \geq 0 \;\vee\; \mathrm{x} + \mathrm{y} \geq 0$
    d. $y > x \;\vee\; \mathrm{x} = 2$

6. Graph the following inequalities:

   a. $y > 2x^2 \wedge \mathrm{x} > 0$

   b. $y > 2x^2 \ \vee \ \mathrm{x} > -1$

   c. $y > 2x^2 \ \vee \ \neg \mathrm{x} > -1$

   d. $y > \neg 2x^2 \ \vee \ \neg \mathrm{x} > -1$

7. Use the Lagrangian to solve the following maximization problems:

   a. $\max_{x_1,x_2}\{f(x_1,x_2) = x_1^2 + 4x_2^2 : x_1 + x_2 = 1\}$

   b. $\max_{x_1,x_2}\{f(x_1,x_2) = x_1 x_2 : x_1^2 + x_2^2 = 8\}$

8. Consider the multivariate function:

$$f(t, x_1(t), x_2(t), \ldots, x_k(t))$$

   a. Take the derivative of this function with respect to t

   b. Derive the formula for the differential *df*.

   c. Calculate the total derivative of the following functions with respect to t.

      1. $y = f(x_1, x_2)$

      2. $y = g(f(x_1, x_2))$

      3. $y = g(f(x_1, x_2), x_2)$

      4. $y = f(x_1, x_2) + g(x_1, x_2)$

Producer Theory

1. Suppose that a firm's production possibilities set is given by the following:

$$Y = \{(y, -x_1, -x_2): y \leq x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} \quad y \geq 0, x_i \geq 0, i = 1,2\}$$

Where y is produced from goods x_1 and x_2.

Calculation Questions

a) Given this production function, if a firm uses 8 units of x_1 and 1 unit of x_2, can the firm produce 3 units of y?

b) If a firm has produced 4 units of output while using 27 units of good x_1, how many units of good x_2 must it have used?

c) What is the maximum the firm can produce given 8 units of x_1 and 8 units of x_2?

d) Can the firm produce 10 units of output given 27 units of x_1 and 8 unit of x_2? What if they are given 8 unit of x_1 and 27 units of x_2?

Comprehension questions

e) Suppose that your firm has committed to produce an output of 10 units under contract for a client. However, there is a shortage of good x_1 which means that the maximum amount available is 64 units. Can your firm honour its production contract?

f) As part of the firm's legal team, you are asked to review a contract which specifies that the firm must supply a minimum output of 10 units of y to fulfill the contract. However, you are fairly certain that there will be a supply shortage of good x_1 during the contract's term. What, if anything, do you advise to add as a contract clause so that the firm can always meet its contractual obligations? Please explain.

g) As the Production Operations Manager for your firm, one of your employment objectives is to make sure that the firm does not waste any of either good x_1 or good x_2 in its output. What new company policy to you set for your plant operator to make sure you always meet your objective?

h) As Chief Operating Officer, you have asked your R&D team to see if they can reduce your firm's reliance on input good x_1. Your investment in research has been a success! Your team tells you that they have come up with a way to produce the same amount of output y, but are now able to use one half of the previous amount of input x_1. The technology does not affect the amount necessary of input x_2. The executive team wants to know whether implementing this technology is a good idea. Please explain what you would tell them.

(Hint: What would the new production function be? Consider how the input requirement sets relate to one another.)

i) Suppose instead that the technology suggested by your R&D team had the same benefit of reducing the required input of good x_1 by one half, but also required an increase in use of x_2 by a factor of 10% at all input levels. Your executive team wants to know whether implementing this technology is a good idea. Please explain what you would tell them.

2. Consider the production possibilities set:

$$Y = \left\{(y, -x) : y \leq (-x)^{1/3}\right\}.$$

a) Graph the set.

b) Is this a valid production possibilities set? If not, what restrictions are necessary on y and x to make it a valid production possibilities set.

c) Recall the definition of free disposal: $\in Y \Rightarrow y' \in Y \; \forall \; y' \leq y$. Does the aforementioned production possibilities set Y satisfy this condition?

d) Recall the definition of convexity $\forall y, y' \in Y, ty + (1-t)y' \in Y$ for $t \in [0,1]$. Does the aforementioned production possibilities set Y satisfy this condition?

3. Consider the following two plants, each of which have different production functions given by:

$$Plant\ A\colon\ Y_A = \{(y, x_1, x_2, x_3)\colon y \leq x_1^{\frac{1}{4}} x_2^{\frac{1}{2}} x_3^{\frac{1}{4}}\ ;\ y \geq 0\ x_i \geq 0, i = 1,2,3\}$$

$$Plant\ B\colon Y_B = \{(y, x_1, x_2, x_3)\colon y \leq x_1^{\frac{1}{3}} x_2^{\frac{1}{2}} x_3^{\frac{1}{3}}, y \geq 0\ x_i \geq 0, i = 1,2,3\}$$

Calculation Questions

a) Given a 1 unit each of goods x_1 and x_3 and 4 units of good x_2, how much will Plant A produce? How much will Plant B produce?

b) Given 2 units each of goods x_1 and x_3 and 1 unit of good x_2, how much will Plant A produce? How much will Plant B produce?

c) Given 2 units each of goods x_1 and x_3 and 2 unit of good x_2, how much will Plant A produce? How much will Plant B produce?

d) Suppose good x_2 is not available. What is the minimum amount of goods x_1 and x_3 required to produce one unit of output?

e) Express mathematically the minimum amount of goods x_1, x_2 and x_3 required to produce one unit of output for both plants.

Comprehension Questions

f) Your company's executive has asked you to make a recommendation for the purchase of a new plant. You have the option of either acquiring plant A or plant B for the same price. Which do you recommend to the executive to purchase? What reason do you give them for your decision?

g) If Plant B is more expensive than Plant A, what do you recommend? Do you need any additional information to make a decision? Please explain.

h) If Plant A is more expensive than Plant B, what do you recommend? Do you need any additional information to make a decision? Please explain.

4. Consider the following production possibilities set for the Leontief technology:

$$Y = \{(y, x_1, x_2): y \leq \min(ax_1, bx_2)\,; y \geq 0, x_i \geq 0, i = 1,2\}$$

This technology matches phone lines (x_1) to call centre employees (x_2) one-to-one for each hour that the call centre is open. Output is given in terms of total hours of dialling by all staff members within that timeframe, termed "calling-hours".

Calculation Questions

a) What are the values of *a* and *b*?

b) Given two telephone lines (x_1) and two call centre staff members (x_2), what is the firm's maximum number of calling-hours? The minimum?

c) Given four telephone lines (x_1) and two call centre staff members (x_2), what is the firm's maximum number of calling-hours?

d) Given one telephone line (x_1) and five call centre staff members (x_2), what is the firm's maximum number of calling-hours?

e) Suppose you have four telephone lines. How many call centre staff members should you schedule so that there is no wasted time or telephone lines?

f) Suppose that you purchase VOIP technology which allows each line to be used concurrently by four call centre staff members. How does this change your answers to parts a) to e)?

Comprehension Questions

i) As the Call Centre Manager, you are asked to complete a project which requires 200 calling-hours. Assuming 20 telephone lines (without VOIP) and eight hour daily shifts, how many call centre staff members will you need to schedule? Workers must work the entire 8 hour shift if they are scheduled. How long will it take you to complete the project?

j) Suppose that the client requires the project to be completed within one eight-hour day. What would you tell the Project Manager that you needed in order to meet the deadline?

k) With a total of 25 telephone lines and a workforce of 20 call centre staff members who each work 40 hours per week, your call centre has 600 calling-hours of business already booked per week on an long-term project. The Director of Sales tells you some great news – that you have a new project coming in! The new project requires another 1200 calling-hours on top of the business you already have booked. Can you complete the project without hiring any new call centre staff or adding any new telephone lines? If so, how long will it take you? If not, how many new staff members do you need in order to complete the project?

5. Consider the Leontief technology for a firm given by the following:

$$Leontief: Y = \{(y, x_1, x_2): y \leq \min(x_1, x_2)\,; y \geq 0, x_i \geq 0, i = 1,2\}$$

An alternative way to think about the minimum value function is by using the sign (sometimes called signum) function, $sgn(x)$. The sign function is defined as follows:

$$sgn(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

a) Graph the functions $f(x) = sgn(x)$ and$f(x) = x\ sgn(x)$

b) Graph the functions $f(x) = x - x\ sgn(x)$ and $f(x) = x + x\ sgn(x)$

c) Determine the values of a, b, c, u, v, w for the following functions

$$f(x) = \begin{cases} a, & b \leq c \\ u, & v > w \end{cases}$$

i) $f(x) = x + x\ sgn(x)$
ii) $f(x) = x\ sgn(x)$
iii) $f(x) = g\big(x - x\ sgn(x)\big)$
iv) $f(x) = g \cdot sgn(x) + h$
v) $f(x) = gx \cdot sgn(x + h)$
vi) $f(x_1, x_2) = sgn(x_1 - x_2)$

Consider the following alternate specification of the Leontief technology:

$$y \leq \min(x_1, x_2) = \begin{cases} x_1, & x_1 \leq x_2 \\ x_2, & x_1 > x_2 \end{cases}$$

d) Express the basic Leontief technology specified above using the sign function.

6. Consider the following specification of Leontief technology for firm A and the Cobb-Douglas technology for firm B:

$$Firm\ A\ (Leontief): Y_A = \{(y, x_1, x_2): y \leq \min(x_1, x_2)\,; y \geq 0, x_i \geq 0, i = 1,2\}$$

$$Firm\ B: (Cobb - Douglas): Y_B = \{(y, x_1, x_2): y \leq x_1^{\frac{1}{n}} x_2^{\frac{1}{n}}; y \geq 0, x_i \geq 0, i = 1,2\}$$

   a) Let n=2.

      i. Calculate the maximum output of the two firms, one using the Leontief technology and the other using the Cobb-Douglas technology, if each uses 1 unit of good x_1 and 1 unit of x_2.

      ii. Calculate the output of the two firms if each uses 16 units of good x_1 and 1 unit of x_2.

      iii. Calculate the output of the two firms if each uses 64 units of good x_1 and 1 unit of x_2.

   b) Calculate question a) for n=4, n=8, n=100, n=1000.

   c) What do you notice about the calculations for parts a) and b)?

7. Suppose you are the CEO of a management consultancy. The company hires junior and senior analysts to produce reports for clients which contain recommendations for their businesses. Because of their different levels of experience, junior analysts make one half of the wage of senior analysts and work the same number of hours. However, due to their higher level of experience, senior analysts also take less time to produce a report. In fact, you have found that, on average, a senior analyst takes 80 hours to write a report, while a junior analyst takes 120 hours.

   At present, your firm produces a total of 40 reports per quarter and charges $15,000 per report.

   There are 2080 work hours per full time employee per year, and therefore 520 work hours per employee per quarter. Assume employees spend 75% of their time on project work (report writing, in this case) and 25% on overhead (administrative tasks which are not billable to a particular project such as meetings and training). All employees must be hired full time but multiple employees can work on the same report.

   a) What is the technical rate of substitution between junior analysts and senior analysts?

   b) If all of the reports were produced by senior analysts, how many would you need to have on staff?

   c) If all the reports were produced by junior analysts, how many would you need to have on staff?

   d) Verify that the TRS is the same as that you calculated in part a) for 40 reports.

e) In what way does the requirement that all employees must be hired full time impact the effective technical rate of substitution? Hint: Consider the case where the company requires an additional three reports to be written for a total of 43 reports.

f) If junior analysts are paid $20 per hour and senior analysts are paid $40 per hour, which of a) or b) has the lower cost?

g) Considering only wage costs:

   i. What ratio of senior to junior analysts should you hire?

   ii. Is this a realistic way to staff your company? Please explain.

h) What functional form will the production function take?

i) Write out the production function. Comment on the substitution between junior and senior analysts.

j) Determine whether the company can meet its workload with the following staffing levels:

   i. 4 junior analysts and 5 senior analysts

   ii. 7 junior analysts and 4 senior analysts

   iii. 3 junior analysts and 7 senior analysts

k) For the feasible staffing levels in part j), what is the total wage cost?

In addition to your wage costs, you must pay overhead for rent and support staff, each of which are related to the total number of staff members you have, irrespective of whether they are senior or junior staff members. Overhead costs are $60 per hour per employee, regardless of whether the employee is spending time on project work or on administrative tasks.

l) Calculate the total costs to your company for staff as described in parts a) and b) and those staffing levels in part j) that are feasible

   i. Does that change the ratio of senior to junior analysts that you should hire? Please explain.

   ii. Is this a realistic way to staff your company?

m) What is the new production function?

n) Calculate the quarterly profit of the firm for hiring all junior analysts, all senior analysts and the feasible staffing levels in part j)

o) Without calculating it, what kind of solution do you expect for a profit maximization problem?

p) Verify your intuition for part o).

q) Why is the analysis of this business inadequate? Hint: What assumptions underlie production theory that may not be met?

8. Consider the following input requirement set:

$$Y = \{(y, -x_1, -x_2): y \leq \ln x_1 + \ln x_2 \ ; y > 0, x_i > 0, i = 1,2\}$$

a) What is the firm's profit maximization problem if its output has price p and inputs x_1 and x_2 cost w_1 and w_2, respectively?

b) What are the first order conditions for the solution?

c) What is the firm's profit as a function of input and output prices?

d) Verify that the profit function satisfy the following properties:

i. Increasing in p

ii. Decreasing in w

iii. Homogeneous of degree one in (p,w)

iv. Satisfies Hotelling's Lemma

9. Consider the following sets:

$$Y_a = \{(y_1, y_2) : y_2 \leq e^{-y_1} - a \lor y_1 \leq 0\}$$

$$Y_b = \{(y_1, y_2) : y_2 \leq e^{-y_1} - a \land y_1 \leq 0\}$$

$$Y_c = \{(y_1, y_2) : y_2 \leq e^{-y_1} - a \lor \neg y_1 \leq 0\}$$

$$Y_d = \{(y_1, y_2) : y_2 \leq e^{-y_1} - a \land \neg y_1 \leq 0\}$$

The logical operators:	AND	OR	NOT
Are expressed by the symbols:	∧ &&	∨ ||	¬ ~ !

a) Graph each of the four sets.

b) Are there values of *a* which make them legitimate production possibilities sets?

10. Consider the following input requirement set:

$$Y = \{(y, -x_1, -x_2): y \leq \ln x_1 + \ln x_2 \;\; ; y > 0, x_i > 0, i = 1,2\}$$

a) What is the firm's profit maximization problem if its output has price p and inputs x_1 and x_2 cost w_1 and w_2, respectively?

b) What are the first order conditions for the solution?

c) What is the firm's efficient output as a function of input and output prices?

d) What is the firm's profit as a function of input and output prices?

11. Consider the following production possibilities set used by a firm:

$$Y = \{(y_1, y_2) : y_2 \leq \sqrt{-y_1}\,\}$$

a) Plot the above inequality.

b) What are the efficient points of the firm's production possibilities set?

c) Show whether or not this production possibilities set is:

   i. Convex

   ii. Closed

   iii. Satisfies free disposal

d) Which of y_1 and y_2 is an input and which is an output?

e) What is the firm's efficient production function in terms of y, the output, and x the input?

f) Write down the profit maximization problem.

g) Derive the first order conditions and solve for the factor demand.

h) What is the output in terms of p and w?

i) What is the profit in terms of p and w?

j) Confirm that if the input and output prices doubled, that the profit would also double. Show generally that the profit function is homogeneous of degree 1 in p and w.

Suppose $(p, w) = (2,5)$.

k) What is the demand at these prices?

l) How much is produced?

m) What is the cost of production?

n) How much revenue is generated?

o) What is the profit?

Suppose that the price of the input decreases to $w = 4$ so that the price vector is $(p, w) = (2,4)$ but that the firm does not change its production.

p) Intuitively, what will happen to the firm's profit? Calculate the profit generated by the firm and verify that your intuition is correct.

q) If the firm were to adjust its production upon seeing the new prices, would the profit be higher or lower than your part (o) answer? Calculate the profit at these prices and verify that your intuition is correct.

Suppose that the price of the firm's output increases to p=3 so that the price vector is $(p, w) = (3,5)$ but that the firm does not change its production.

r) Intuitively, what will happen to the firm's profit? Calculate the profit generated by the firm and verify that your intuition is correct.

s) If the firm were to adjust its production upon seeing the new prices, would the profit be higher or lower than your part (r) answer? Calculate the profit at these prices and verify that your intuition is correct.

t) Examine the profit with and without readjustment when:

    i. The input price decreases to w=1

    ii. The output price decreases to p=1

    iii. Both input and output prices simultaneously increase. Can you determine whether the profit will increase or decrease intuitively? Why or why not?

u) Consider two price vectors, $(p, w) = (2,2)$ and $(p, w) = (4,6)$.

    i. Calculate the optimal profit at each of these points

    ii. Calculate the average profit of these two price vectors.

    iii. Intuitively, should the profit at the average of these two prices, ie: at $(p, w) = (3,4)$ be higher or lower than your part ii) answer? Calculate the profit at (3,4) and verify your intuition is right.

12. Consider the following production possibilities set used by a firm:

$$Y = \{(y_1, y_2) : (y_1 - 1)(y_2 - 1) \geq 1 \wedge y_1 \leq 0\}$$

a) Plot the above inequality.
b) What are the efficient points of the firm's production possibilities set?
c) Show whether or not this production possibilities set is:
    i. Convex
    ii. Closed
    iii. Satisfies free disposal
d) Which of y_1 and y_2 is an input and which is an output?
e) What is the firm's efficient production function in terms of y, the output, and x the input?
f) Write down the profit maximization problem.
g) Derive the first order conditions and solve for the factor demand.
h) What is the output in terms of p and w?
i) What is the profit in terms of p and w?
j) Confirm that if the input and output prices doubled, that the profit would also double. Show generally that the profit function is homogeneous of degree 1 in p and w. Intuitively, what does this mean? Has the profit really doubled? Why or why not?

Suppose $(p, w) = (4,4)$.

k) What is the demand at these prices, how much is produced and what is the profit?
l) What is the value of the conditional factor demand, the output and profit when p=w?

Suppose (p,w)=(9,4).

m) What is the demand at these prices?
n) How much is produced?
o) What is the cost of production?
p) How much revenue is generated?
q) What is the profit?
r) Suppose that the price of its output increases to p=16 so that the price vector is (p,w)=(16,4). However, the firm does not change its production. Intuitively, what will happen to the firm's profit? Calculate the profit generated by the firm and verify that your intuition is correct.
s) If the firm were to adjust its production upon seeing the new prices, would the profit be higher or lower than your part (m) answer? Calculate the profit at these prices and verify that your intuition is correct.

Consumer Theory

1. What is the <u>most general axiom</u> necessary for the following statements to hold?

    a) Any bundle is at least as good as itself.
    b) Upper contour sets do not have thick boundaries.
    c) At least as much of each good is at least as good.
    d) A consumer prefers greater variety over less
    e) If one bundle, x_a is at least as good as another bundle, x_b, then any bundle at least as good as x_a is also at least as good as x_b.
    f) A better bundle can always be found no matter how close you are to the initial bundle.
    g) Any two bundles can be compared and one found to be at least as good as the other.
    h) More of any one good is better.

2. Suppose the consumer's utility is given by the following utility function:

$$u = \sqrt{x_1 x_2 + x_1}$$

    a) What is the consumer's utility maximization problem?
    b) What are the consumer's factor demands for goods (x_1, x_2)?
    c) What are the consumer's factor demands at the following price and budget levels:
        i. $\boldsymbol{p} = (p_1, p_2) = (1, 4); m = 10$
        ii. $\boldsymbol{p} = (p_1, p_2) = (1, 4); m = 20$
        iii. $\boldsymbol{p} = (p_1, p_2) = (1, 2); m = 10$
        iv. $\boldsymbol{p} = (p_1, p_2) = (1, 2); m = 20$
    d) What is the consumer's indirect utility function?
    e) Verify that the consumer's indirect utility function satisfies the following properties:
        i. Non-increasing in p
        ii. Non-decreasing in m
        iii. Satisfies Roy's Identity
        iv. Homogeneous of degree 0 in (p_1, p_2, m)
    f) The condition $(m - p_2) > 0$ was mentioned in passing. What happens if $p_2 > m$?
    g) What is the consumer's expenditure function?
    h) Verify that the following expenditure function is non-decreasing in p
    i) Verify that the following expenditure function is homogeneous of degree one in p
    j) Derive the Hicksian Demands for the consumer using the identity:

    $\boldsymbol{h}(\boldsymbol{p}, u) \equiv \boldsymbol{x}(\boldsymbol{p}, e(\boldsymbol{p}, u))$

    k) Verify that the Hicksian demands satisfy Shephard's Lemma.

3. Suppose the consumer's utility is given by the following utility function:

$$u = -\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2}$$

a) What is the consumer's utility maximization problem?
b) Solve the consumer's utility maximization problem by setting up the Lagrangian.
c) If $\boldsymbol{p} = (p_1, p_2) = (1, 1)$ $and\ m = 8$ what are the consumer's factor demands?
d) What is the consumer's utility at this level of demand?
e) Is the bundle $\boldsymbol{x} = (x_1, x_2) = (2,4)$ affordable at $\boldsymbol{p} = (p_1, p_2) = (1, 1)$?
f) What is the consumer's utility at the point $= (x_1, x_2) = (2,4)$?
g) Compare the consumer's utility from part f) and part d). What do you notice?
h) Why did the normal procedure fail? Hint: Consider whether all of the axioms are indeed met. Specifically examine the point $\boldsymbol{x} = (x_1, x_2) = (2,4)$

4. Suppose the following information is collected about the behaviour of a consumer in three time periods.

	Time Period 1	Time Period 2	Time Period 3
Prices	$\boldsymbol{p}^1 = (2,2)$	$\boldsymbol{p}^2 = (1,2)$	$\boldsymbol{p}^3 = (2,3)$
Person A's Choice	$\boldsymbol{x}_A^1 = (2,2)$	$\boldsymbol{x}_A^2 = (2,1)$	$\boldsymbol{x}_A^3 = (3,2)$

a) What is Person A's expenditure in period 1?
b) Could have Person A afforded the bundles he consumed in periods 2 and 3?
c) Can you draw any conclusions on which bundle is preferred based on your a) and b) part answer?
d) Repeat parts a) through c) for bundles 2 and 3.
e) Do the preferences of Person A satisfy WARP?

5. Suppose that a consumer is given an endowment $\omega = (\omega_1, \omega_2, \omega_3) = (2,1,3)$

This consumer's utility function is given by:

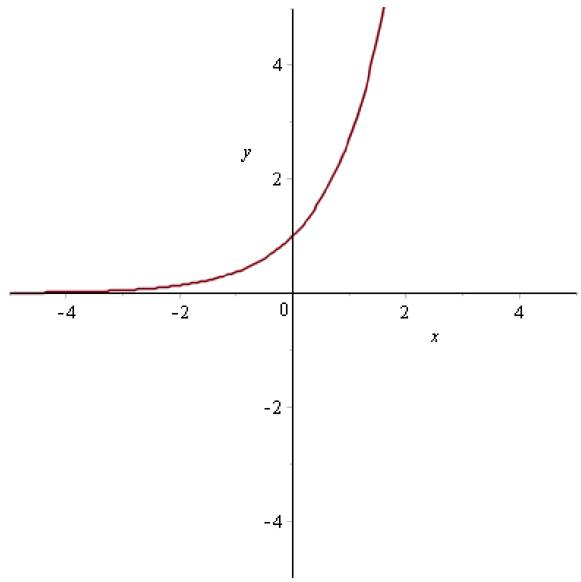
$$u(x_1, x_2, x_3) = x_1 x_2 + x_3$$

a) What is the consumer's utility with their initial endowment?
b) What is the consumer's maximization problem?
c) Suppose that the consumer was faced with prices $p = (p_1, p_2, p_3) = (1,1,4)$. What is the consumer's budget constraint?
d) What are the consumer's demands at these prices?
e) Are the demands of each good higher or lower than the endowment?
f) Verify that the consumer's utility is higher after the exchange.

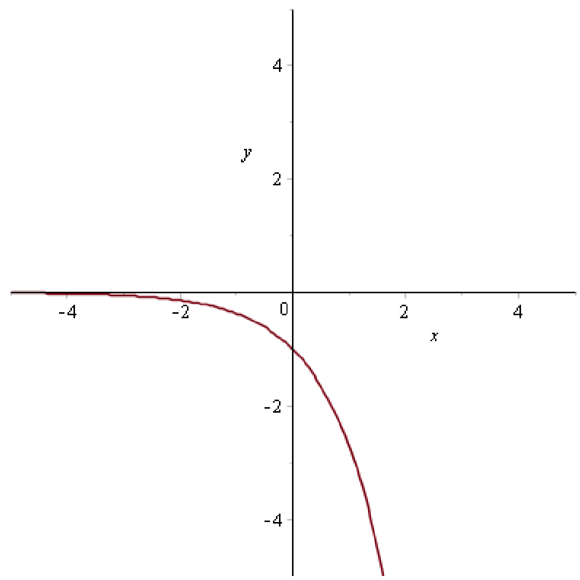
Solutions Mathematical Preliminaries

1. Graph the following family of functions:

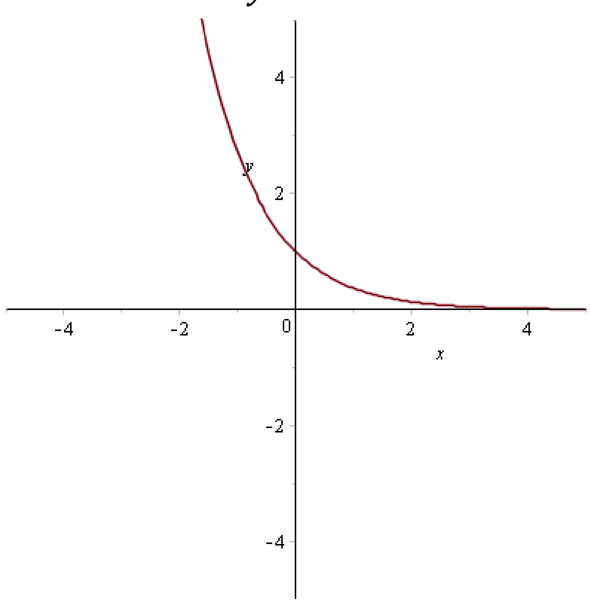
a. $y = e^{x}$



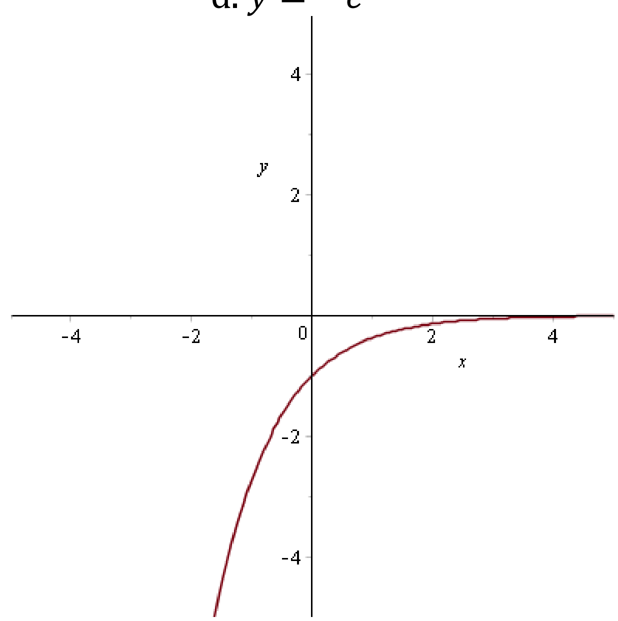
b. $y = -e^{x}$



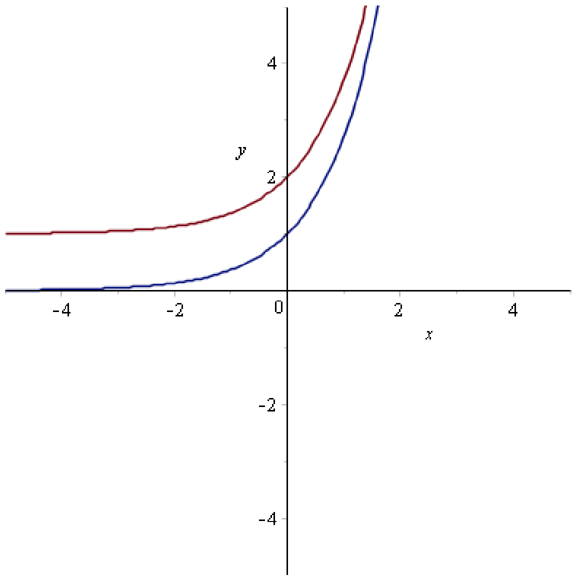
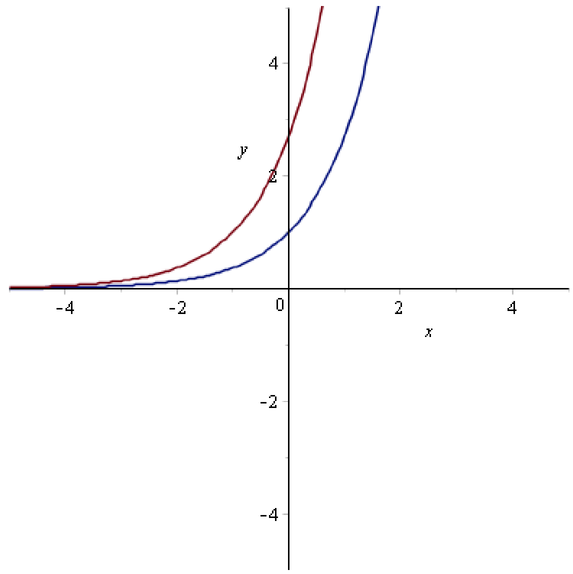
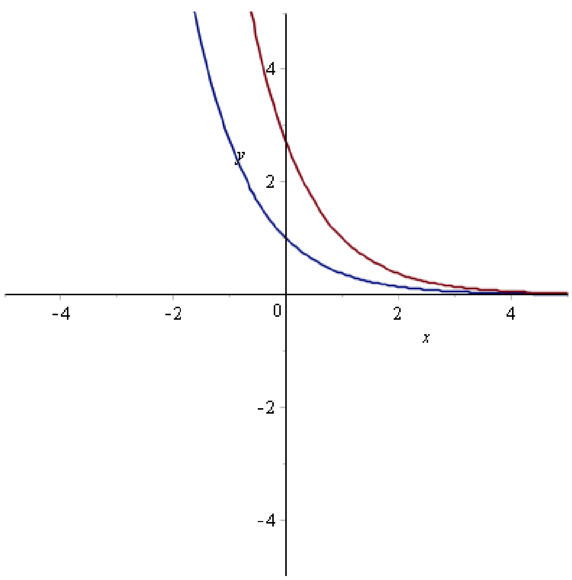
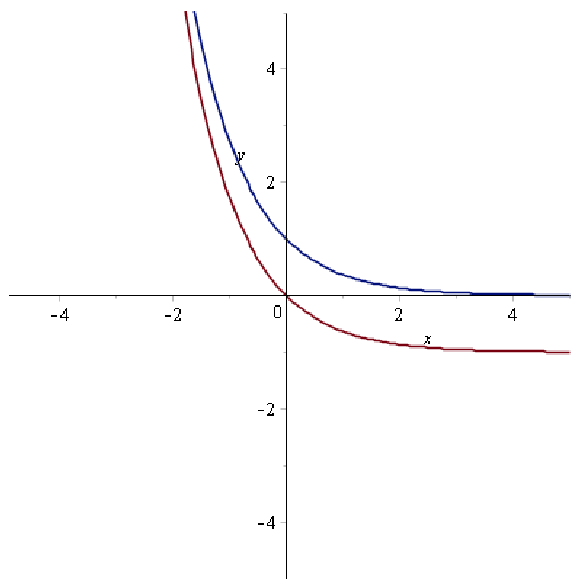
c. $y = e^{-x}$



d. $y = -e^{-x}$

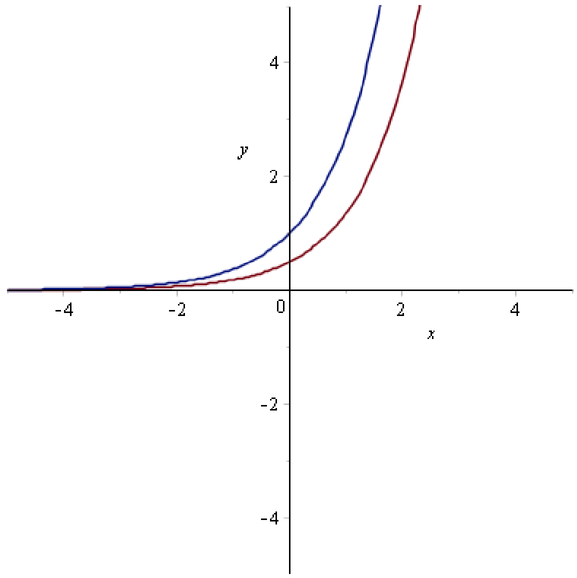


2. Graph the following family of functions:

a. $y = e^x + 1$  The graph of e^x is shown in blue.	b. $y = e^{x+1}$  The graph of e^x is shown in blue
c. $y = e^{-x}$  The graph of e^{-x} is shown in blue.	d. $y = -e^{-x} - 1$  The graph of e^{-x} is shown in blue.

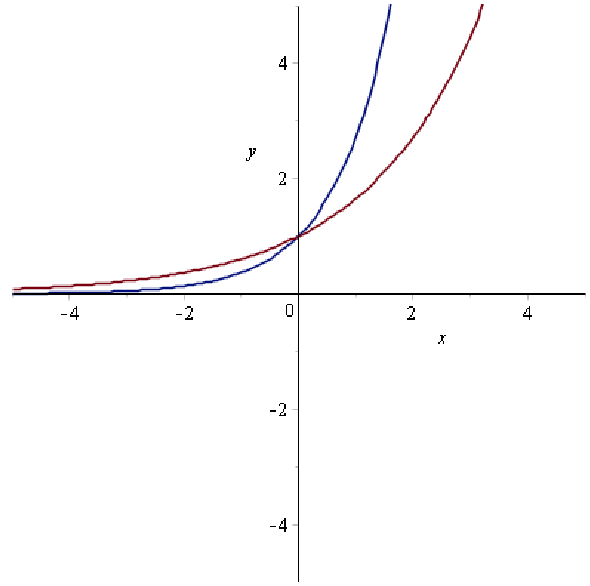
3. Graph the following family of functions:

a. $y = \frac{e^x}{2}$



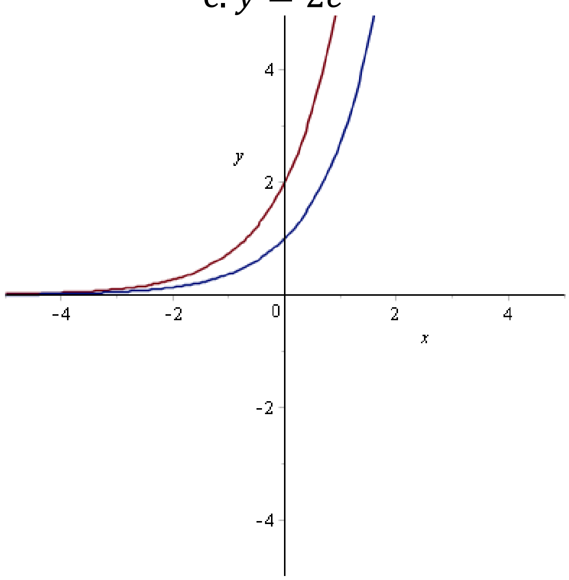
The graph of e^x is shown in blue.

b. $y = e^{x/2}$



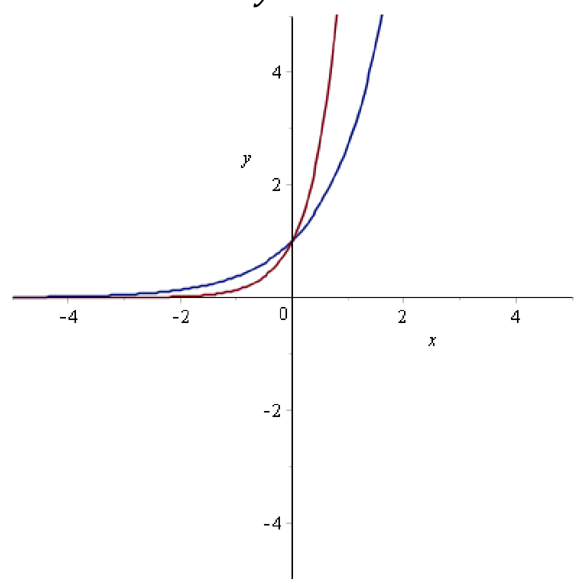
The graph of e^x is shown in blue

c. $y = 2e^x$



The graph of e^x is shown in blue.

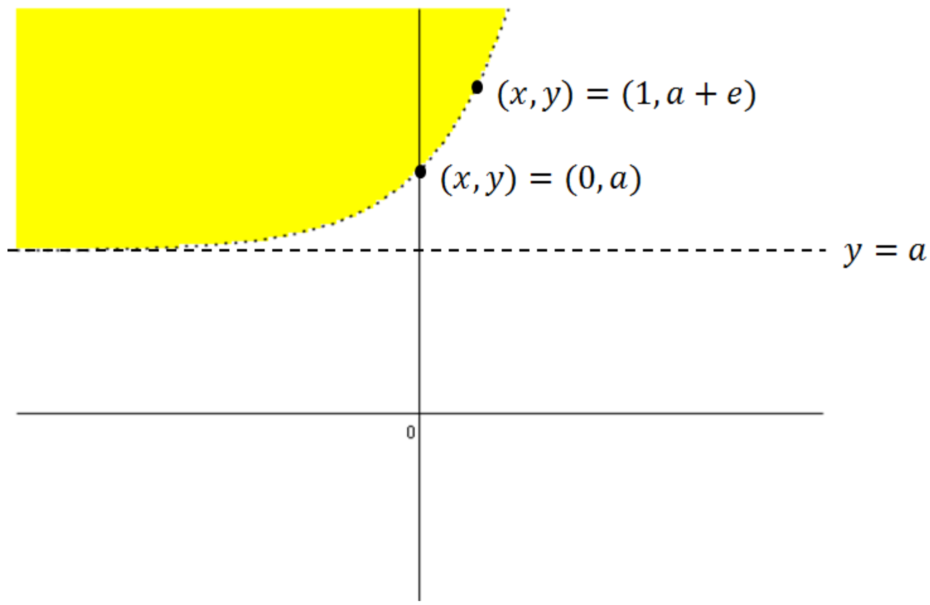
d. $y = e^{2x}$



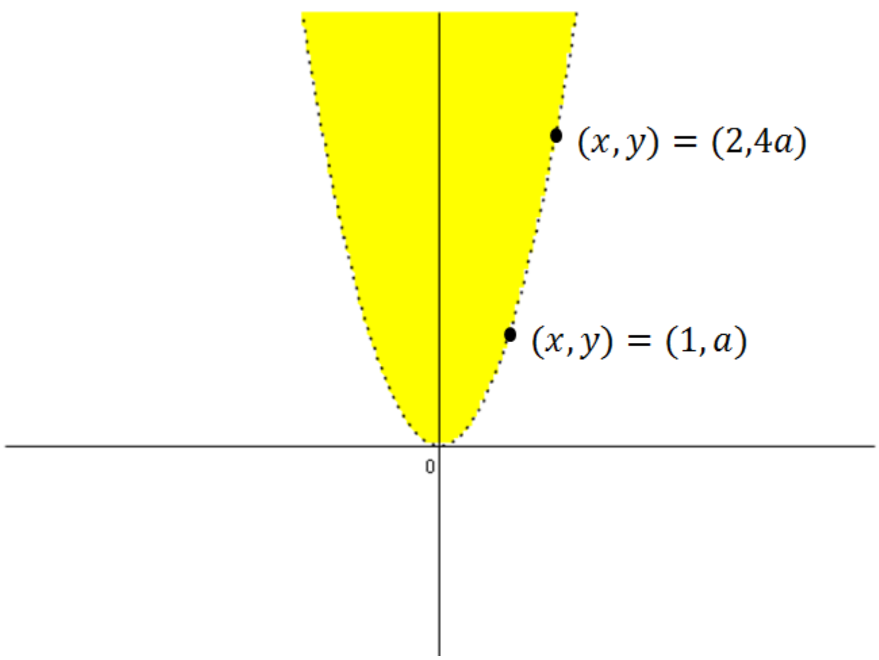
The graph of e^x is shown in blue.

4. Graph the following:

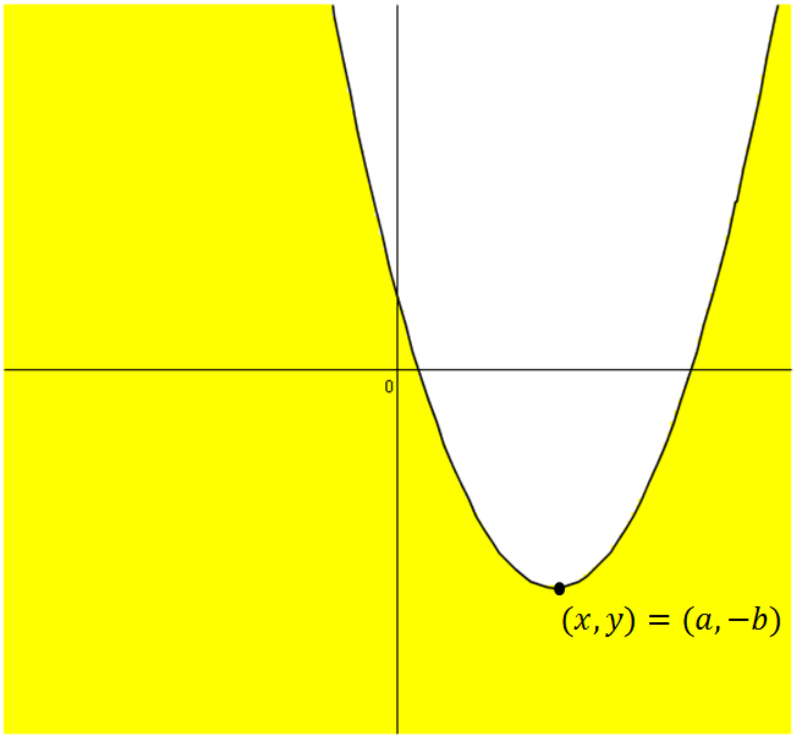
   a. $y < e^x + a \; where \; a > 0$



   b. $y > ax^2 where \; a > 0$

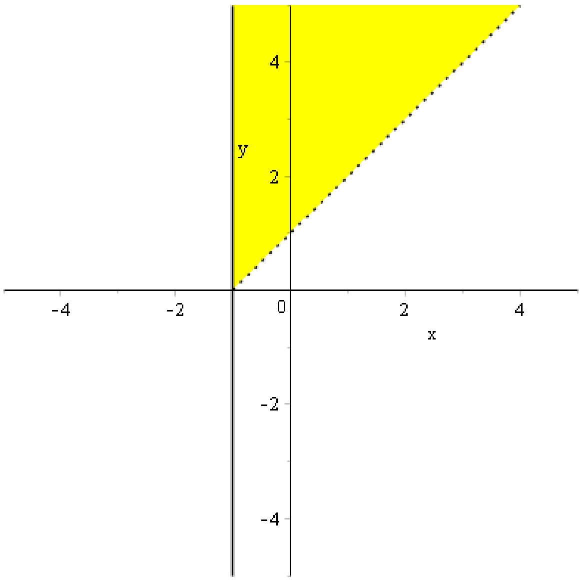


c. $y \leq (x-a)^2 - b$ *where* $a > 0; \mathrm{b} > 0$

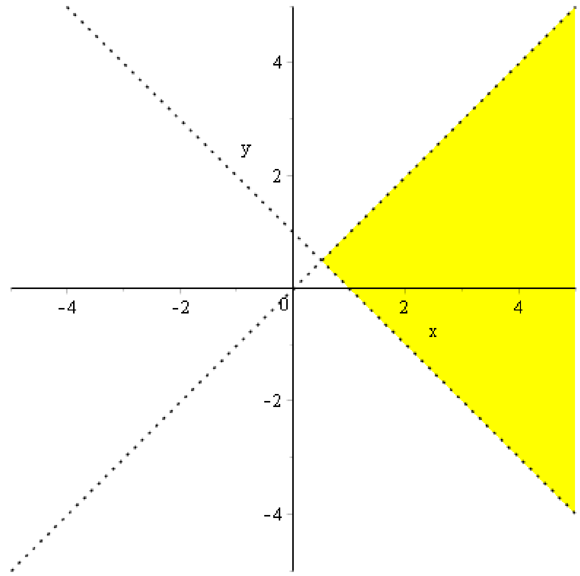


5. Graph the following inequalities:

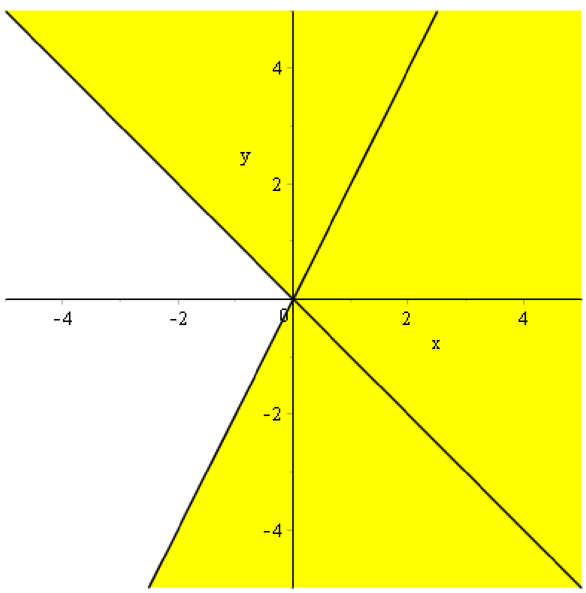
a. $y > x + 1 \ \wedge \ x \geq -1$



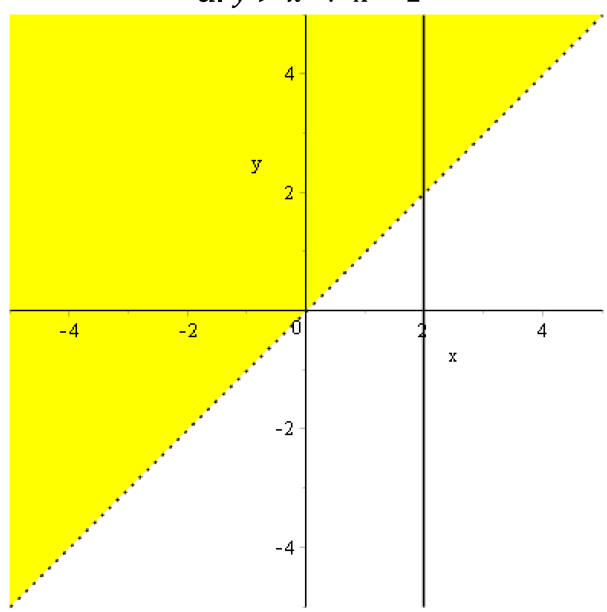
b. $x + y > 1 \ \wedge \ x - y > 0$



c. $2x - y \geq 0 \ \vee \ x + y \geq 0$

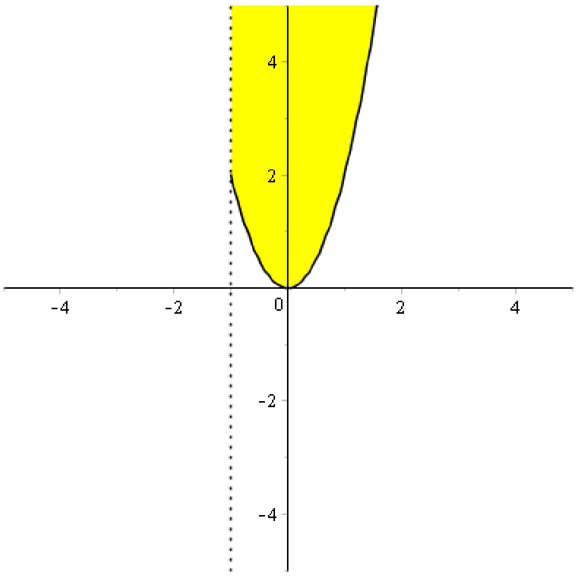


d. $y > x \ \vee \ x = 2$

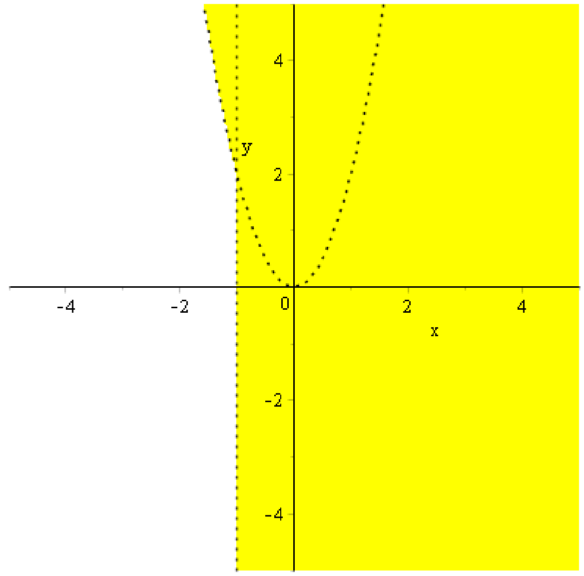


6. Graph the following inequalities:

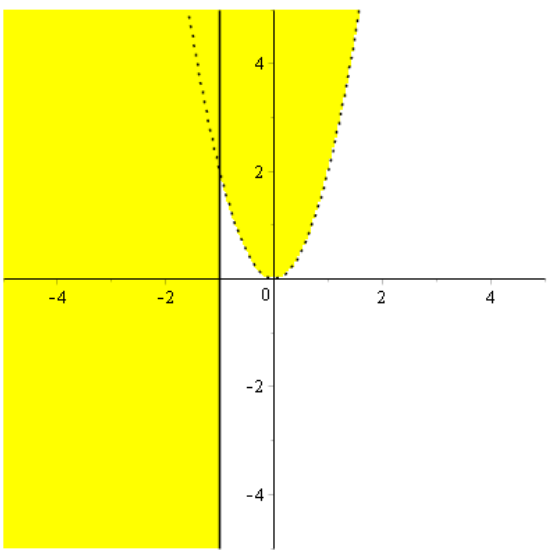
a. $y > 2x^2 \wedge x > 0$



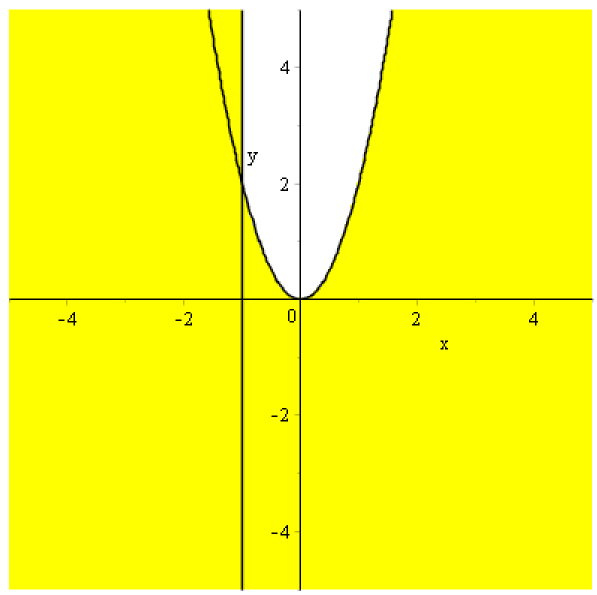
b. $y > 2x^2 \ \vee \ x > -1$



c. $y > 2x^2 \ \vee \ \neg x > -1$



d. $y > \neg 2x^2 \ \vee \ \neg x > -1$



7. Use the Lagrangian to solve the following maximization problems:

a. $\max_{x_1,x_2}\{f(x_1,x_2) = x_1^2 + 4x_2^2 : x_1 + x_2 = 1\}$

The Lagrangian is:

$$L = x_1^2 + 4x_2^2 + \lambda(x_1 + x_2 - 1)$$

And therefore, the first order conditions are:

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2\lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 8x_2 + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 - 1 = 0$$

Therefore,

$$x_1 = -\lambda$$

$$4x_2 = -\lambda$$

$$x_1 = 4x_2$$

$$4x_2 + x_2 = 1$$

$$5x_2 = 1$$

$$x_2 = \frac{1}{5}$$

$$x_1 = \frac{4}{5}$$

b. $\max_{x_1,x_2}\{f(x_1,x_2) = x_1x_2 : x_1^2 + x_2^2 = 8\}$

The Lagrangian is:

$$L = x_1x_2 + \lambda(x_1^2 + x_2^2 - 8)$$

The first order conditions are:

$$\frac{\partial L}{\partial x_1} = x_2 + 2\lambda x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = x_1 + 2\lambda x_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 - 8 = 0$$

Therefore:

$$x_2 = -2\lambda x_1$$

and

$$x_1 = -2\lambda x_2$$

Combining them gives

$$x_1^2 = x_2^2 \Leftrightarrow |x_1| = |x_2|$$

So, $x_1^2 + x_1^2 - 8 = 0 \Leftrightarrow 2x_1^2 = 8 \Leftrightarrow x_1^2 = 4 \Leftrightarrow x_1 = \pm 2$

Therefore $x_2 = \pm 2$

The solutions are $(x_1, x_2) = (2,2)$ and $(x_1, x_2) = (-2, -2)$.

8. Consider the multivariate function:

$$f(t, x_1(t), x_2(t), \dots, x_k(t))$$

a. Take the derivative of this function with respect to t

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_k}\frac{dx_k}{dt}$$

b. Derive the formula for the differential *df*.

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_k}dx_k$$

c. Calculate the total derivative of the following functions with respect to t.

1. $y = f(x_1, x_2)$

$$\frac{dy}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$$

2. $y = g(f(x_1, x_2))$

$$\frac{dy}{dt} = \frac{dg}{df}\frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{dg}{df}\frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$$

3. $y = g(f(x_1, x_2), x_2)$

$$\frac{dy}{dt} = \frac{dg}{df}\frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{dg}{df}\frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$$

4. $y = f(x_1, x_2) + g(x_1, x_2)$

$$\frac{dy}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial g}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial g}{\partial x_2}\frac{dx_2}{dt}$$

Producer Theory Solutions

1. Suppose that a firm's production possibilities set is given by the following:

$$Y = \{(y, -x_1, -x_2): y \leq x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} \;\; y \geq 0, x_i \geq 0, i = 1,2\}$$

Where y is produced from goods x_1 and x_2.

Calculation Questions

a) Given this production function, if a firm uses 8 units of x_1 and 1 unit of x_2, can the firm produce 3 units of y?

$$y \leq 8^{\frac{1}{3}} 1^{\frac{2}{3}} = 2$$

The maximum that the firm can product with $(x_1, x_2) = (8,1)$ is $y = 2$ units. Therefore they cannot produce 3 units of y.

b) If a firm has produced 4 units of output while using 27 units of good x_1, how many units of good x_2 must it have used?

$$4 \leq 27^{\frac{1}{3}} x_2^{\frac{2}{3}} = 3x_2^{\frac{2}{3}}$$

$$\frac{4}{3} \leq x_2^{\frac{2}{3}}$$

$$\left(\frac{4}{3}\right)^{\frac{3}{2}} \leq x_2$$

$$\frac{8}{\sqrt{27}} = \frac{8}{9}\sqrt{3} \cong 1.5396 \leq x_2$$

The firm must have used *at least* $x_2 = \frac{8}{9}\sqrt{3}$, however it could have used more.

c) What is the maximum the firm can produce given 8 units of x_1 and 8 units of x_2?

$$y \leq 8^{\frac{1}{3}} 8^{\frac{2}{3}} = 2 \cdot 4 = 8$$

The firm can produce at most $y = 8$.

d) Can the firm produce 10 units of output given 27 units of x_1 and 8 unit of x_2? What if they are given 8 unit of x_1 and 27 units of x_2?

$$y \leq 27^{\frac{1}{3}}8^{\frac{2}{3}} = 3 \cdot 4 = 12$$

Yes, the firm can produce 10 units if they are given 27 units of x_1 and 8 units of x_2.

$$y \leq 27^{\frac{2}{3}}8^{\frac{1}{3}} = 9 \cdot 2 = 18$$

Yes, the firm can produce 10 units if they are given 8 units of x_1 and 27 units of x_2.

Comprehension questions

e) Suppose that your firm has committed to produce an output of 10 units under contract for a client. However, there is a shortage of good x_1 which means that the maximum amount available is 64 units. Can your firm honour its production contract?

The firm can honor the contract only if there is at least some of good x_1 available (ie: $x_1 > 0$) and that there is no restriction on the amount of x_2 that is available.

f) As part of the firm's legal team, you are asked to review a contract which specifies that the firm must supply a minimum output of 10 units of y to fulfill the contract. However, you are fairly certain that there will be a supply shortage of good x_1 during the contract's term. What, if anything, do you advise to add as a contract clause so that the firm can always meet its contractual obligations? Please explain.

The contract must specify that there is *at least some* units of x_1 available and that there is *no restriction* on the amount of x_2 that is available.

g) As the Production Operations Manager for your firm, one of your employment objectives is to make sure that the firm does not waste any of either good x_1 or good x_2 in its output. What new company policy to you set for your plant operator to make sure you always meet your objective?

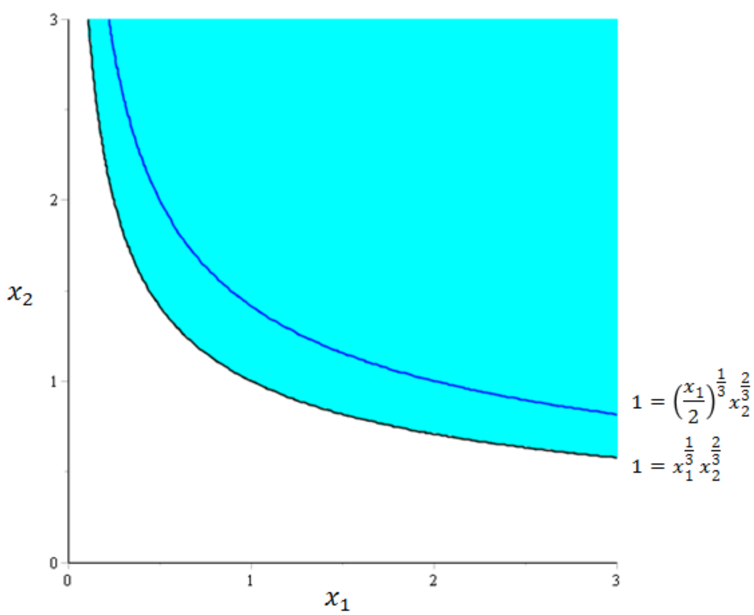
To ensure that the firm does not waste any of good x_1 or x_2, he firm must produce efficiently. Therefore, production should be $y = x_1^{\frac{1}{3}}x_2^{\frac{2}{3}}$.

h) As Chief Operating Officer, you have asked your R&D team to see if they can reduce your firm's reliance on input good x_1. Your investment in research has been a success! Your team tells you that they have come up with a way to produce the same amount of output y, but are now able to use one half of the previous amount of input x_1. The technology does not affect the amount necessary of input x_2. The executive team wants to know whether implementing this technology is a good idea. Please explain what you would tell them. (Hint: What would the new production function be? Consider how the input requirement sets relate to one another.)

The new production possibilities set is

$$Y_{New} = \{(y, -x_1, -x_2): y \leq \left(\frac{x_1}{2}\right)^{\frac{1}{3}} x_2^{\frac{2}{3}} \quad y \geq 0, x_i \geq 0, i = 1,2\}$$

For each level of x_1, the old production possibilities set is completely contained within the new set: Since $x_1 \geq \frac{x_1}{2}$, $x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} \geq \left(\frac{x_1}{2}\right)^{\frac{1}{3}} x_2^{\frac{2}{3}}$. An example of two input requirement sets for $y = 1$ is shown below:



Since this technology allows for increased production for any levels of x_1 and x_2, implementing this technology is a good idea. It allows the firm to produce more with the same levels of inputs compared to the previous technology.

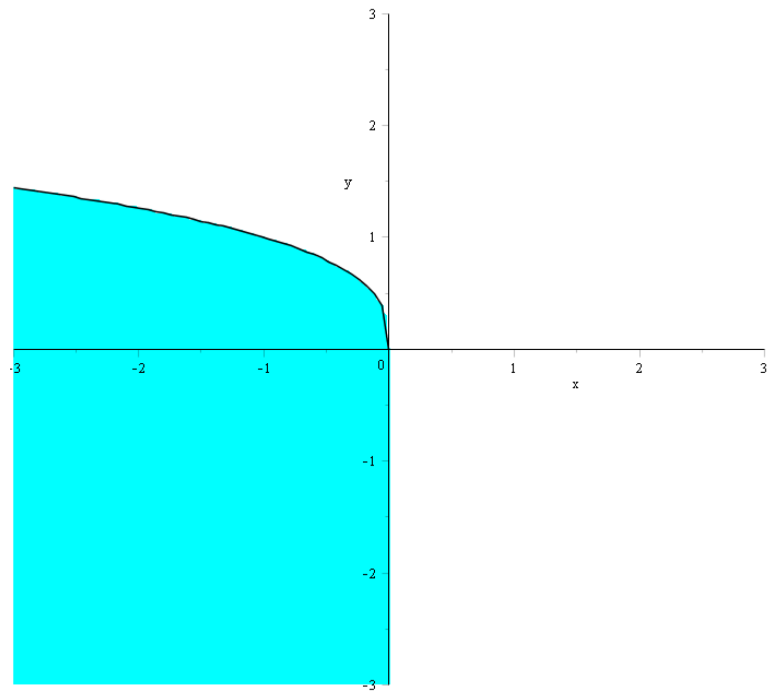
i) Suppose instead that the technology suggested by your R&D team had the same benefit of reducing the required input of good x_1 by one half, but also required an increase in use of x_2 by a factor of 10% at all input levels. Your executive team wants to know whether implementing this technology is a good idea. Please explain what you would tell them.

In this case, the new input requirement sets are not completely contained within the old input requirement sets. The result is ambiguous. Implementing the technology may not be profitable for the firm if the cost of the input x_2 is much higher than the cost of input x_1.

2. Consider the production possibilities set:

$$Y = \left\{(y, -x) : y \leq (-x)^{1/3}\right\}.$$

i. Graph the set.



e) Is this a valid production possibilities set? If not, what restrictions are necessary on y and x to make it a valid production possibilities set.

Yes.

f) Recall the definition of free disposal: $\in Y \Rightarrow y' \in Y \; \forall \, y' \leq y$. Does the aforementioned production possibilities set Y satisfy this condition?

Yes.

g) Recall the definition of convexity $\forall y, y' \in Y, ty + (1-t)y' \in Y$ for $t \in [0,1]$. Does the aforementioned production possibilities set Y satisfy this condition?

Yes.

3. Consider the following two plants, each of which have different production functions given by:

$$Plant\ A\text{: } Y_A = \{(y, x_1, x_2, x_3): y \leq x_1^{\frac{1}{4}} x_2^{\frac{1}{2}} x_3^{\frac{1}{4}}\ ;\ y \geq 0\ x_i \geq 0, i = 1,2,3\}$$

$$Plant\ B\text{: } Y_B = \{(y, x_1, x_2, x_3): y \leq x_1^{\frac{1}{3}} x_2^{\frac{1}{2}} x_3^{\frac{1}{3}}, y \geq 0\ x_i \geq 0, i = 1,2,3\}$$

Calculation Questions

a) Given a 1 unit each of goods x_1 and x_3 and 4 units of good x_2, how much will Plant A produce? How much will Plant B produce?

$$y_A \leq 1^{\frac{1}{4}} \cdot 4^{\frac{1}{2}} \cdot 1^{\frac{1}{4}} = 1 \cdot 2 \cdot 1 = 2$$

$$y_B \leq 1^{\frac{1}{3}} \cdot 4^{\frac{1}{2}} \cdot 1^{\frac{1}{3}} = 1 \cdot 2 \cdot 1 = 2$$

Both Plant A and Plant B produce at most 2 units of y.

b) Given 2 units each of goods x_1 and x_3 and 1 unit of good x_2, how much will Plant A produce? How much will Plant B produce?

$$y_A \leq 2^{\frac{1}{4}} \cdot 1^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} = 2^{\frac{1}{2}} \cong 1.414$$

$$y_B \leq 2^{\frac{1}{3}} \cdot 1^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} = 2^{\frac{2}{3}} \cong 1.587$$

Plant A will produce at most $2^{\frac{1}{2}}$ units and Plant B will produce at most $2^{\frac{2}{3}}$ units.

c) Given 2 units each of goods x_1 and x_3 and 2 unit of good x_2, how much will Plant A produce? How much will Plant B produce?

$$y_A \leq 2^{\frac{1}{4}} \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} = 2$$

$$y_B \leq 2^{\frac{1}{3}} \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} = 2^{\frac{7}{6}} \cong 2.245$$

Plant A will produce at most 2 units and Plant B will produce at most $2^{\frac{7}{6}}$ units.

d) Suppose good x_2 is not available. What is the minimum amount of goods x_1 and x_3 required to produce one unit of output?

If good x_2 is not available, neither plant can produce anything:

$$y_A \leq x_1^{\frac{1}{4}} \cdot 0 \cdot x_3^{\frac{1}{4}} = 0$$

$$y_B \leq x_1^{\frac{1}{3}} \cdot 0 \cdot x_3^{\frac{1}{3}} = 0$$

e) Express mathematically the minimum amount of goods x_1, x_2 and x_3 required to produce one unit of output for both plants.

$$Plant\ A: 1 = x_1^{\frac{1}{4}} x_2^{\frac{1}{2}} x_3^{\frac{1}{4}}$$

$$Plant\ B: 1 = x_1^{\frac{1}{3}} x_2^{\frac{1}{2}} x_3^{\frac{1}{3}}$$

Comprehension Questions

f) Your company's executive has asked you to make a recommendation for the purchase of a new plant. You have the option of either acquiring plant A or plant B for the same price. Which do you recommend to the executive to purchase? What reason do you give them for your decision?

Plant B is the better purchase. For equal input amounts, Plant B can produce more compared to Plant A.

g) If Plant B is more expensive than Plant A, what do you recommend? Do you need any additional information to make a decision? Please explain.

If Plant B is more expensive, then more information is needed. Either Plant A or Plant B could be a better purchase depending on the relative prices of the input and output goods, production levels and prices of the firms.

h) If Plant A is more expensive than Plant B, what do you recommend? Do you need any additional information to make a decision? Please explain.

If Plant A is more expensive, no more information is needed. Plant B is the better purchase, since Plant B is always able to produce more for the same amount of inputs.

4. Consider the following production possibilities set for the Leontief technology:

$$Y = \{(y, x_1, x_2): y \leq \min(ax_1, bx_2)\,; y \geq 0, x_i \geq 0, i = 1,2\}$$

This technology matches phone lines (x_1) to call centre employees (x_2) one-to-one for each hour that the call centre is open. Output is given in terms of total hours of dialling by all staff members within that timeframe, termed "calling-hours".

Calculation Questions

a) What are the values of a and b?

Each employee needs to be matched to a phone line. Therefore, $a = b = 1$.

b) Given two telephone lines (x_1) and two call centre staff members (x_2), what is the firm's maximum number of calling-hours? The minimum?

The maximum number of calling hours is $y = \min(x_1, x_2) = \min(2,2) = 2$. The minimum is zero – the firm does not have to dial at all.

c) Given four telephone lines (x_1) and two call centre staff members (x_2), what is the firm's maximum number of calling-hours?

The maximum number of calling hours is $y = \min(x_1, x_2) = \min(4,2) = 2$. Even though the firm has access to four telephone lines, they do not have enough call centre staff members to utilize them.

d) Given one telephone line (x_1) and five call centre staff members (x_2), what is the firm's maximum number of calling-hours?

The maximum number of calling hours is $y = \min(x_1, x_2) = \min(1,5) = 1$. Even though the firm has five call centre staff members, there are not enough telephone lines.

e) Suppose you have four telephone lines. How many call centre staff members should you schedule so that there is no wasted time or telephone lines?

$$y = \min(4, x_2) = \begin{cases} x_2 \; if \; x_2 < 4 \\ 4 \; if \; x_2 \geq 4 \end{cases}$$

To be efficient, the call centre staff members must be paired to telephone lines one-to-one. Therefore, only if $x_2 = 4$ will each of the telephone lines and call centre staff members utilized.

f) Suppose that you purchase VOIP technology which allows each line to be used concurrently by four call centre staff members. How does this change your answers to parts a) to e)?

a) In this case, four call centre staff members can use a single telephone line. Therefore

$$a = 4; b = 1$$

The maximum number of calling hours is $y = \min(4x_1, x_2)$.

b) The maximum number of calling hours is $y = \min(4 \cdot 2,2) = 2$. The answer is identical to the first case because there are only two staff members.

c) The maximum number of calling hours is $y = \min(4 \cdot 4,2) = 2$. The answer is identical to the first case because there are only two staff members.

d) The maximum number of calling hours is $y = \min(4x_1, x_2) = \min(4,5) = 4$. The answer changes from the first case as four staff members can now use a single telephone line.

e)

$$y = \min(4 \cdot 4, x_2) = \begin{cases} x_2 \; if \; x_2 < 16 \\ 16 \; if \; x_2 \geq 16 \end{cases}$$

To be efficient, four call centre staff members can use a single telephone line. Therefore, only if $x_2 = 16$ will each of the telephone lines and call centre staff members used.

Comprehension Questions

i) As the Call Centre Manager, you are asked to complete a project which requires 200 calling-hours. Assuming 20 telephone lines (without VOIP) and eight hour daily shifts, how many call centre staff members will you need to schedule? Workers must work the entire 8 hour shift if they are scheduled. How long will it take you to complete the project?

To be efficient, 20 call centre staff members should be scheduled. Therefore, each hour will produce $y = \min(20,20) = 20$ calling-hours. This means that it will take $\frac{200}{20} = 10$ hours if all employees are working to complete the project. However, if all employees must work the entire eight hour shift, two days will be needed. In the first day, 20 employees are able to complete $8 \cdot 20 = 160$ calling-hours, leaving 40 hours left of the project. Therefore, the next day, $\frac{40}{8} = 5$ staff members are needed.

j) Suppose that the client requires the project to be completed within one eight-hour day. What would you tell the Project Manager that you needed in order to meet the deadline?

To complete the project, the call centre's capacity must be increased. The restriction on telephone lines means that the project cannot be completed regardless of the staffing level. Therefore, more telephone lines or a VOIP system must be added.

k) With a total of 25 telephone lines and a workforce of 20 call centre staff members who each work 40 hours per week, your call centre has 600 calling-hours of business already booked per week on an long-term project. The Director of Sales tells you some great news – that you have a new project coming in! The new project requires another 1200 calling-hours on top of the business you already have booked. Can you complete the project without hiring any new call centre staff or adding any new telephone lines? If so, how long will it take you? If not, how many new staff members do you need in order to complete the project?

The production function means that the call centre production with the present staffing levels and telephone lines is $y = \min(25,20) = 20$. The weekly capacity if all 20 employees worked full-time for the entire week is therefore $20 \cdot 40 = 800$. Booked business takes up 600 which means that an additional $800 - 600 = 200$ calling-hours per week is available over and above the long-term project. The new project will take $\frac{1200}{200} = 6$ weeks to complete.

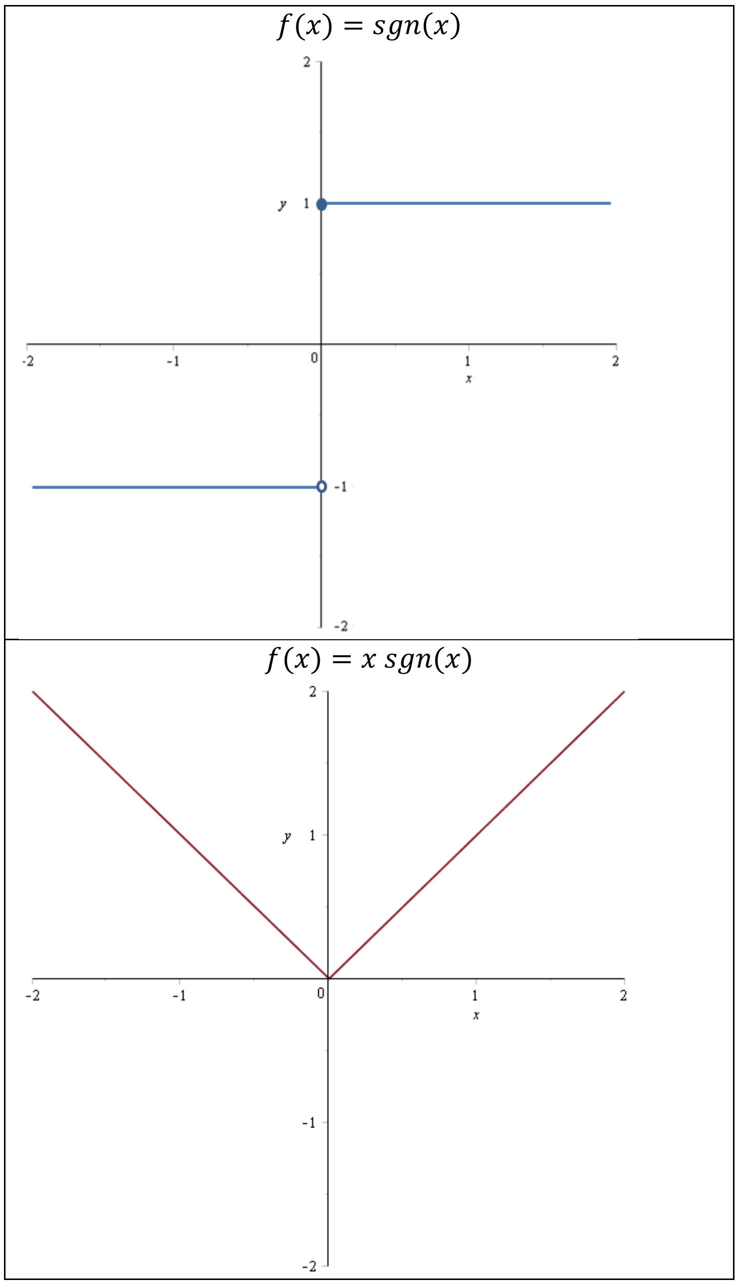
5. Consider the Leontief technology for a firm given by the following:

$$Leontief: Y = \{(y, x_1, x_2): y \leq \min(x_1, x_2)\,; y \geq 0, x_i \geq 0, i = 1,2\}$$

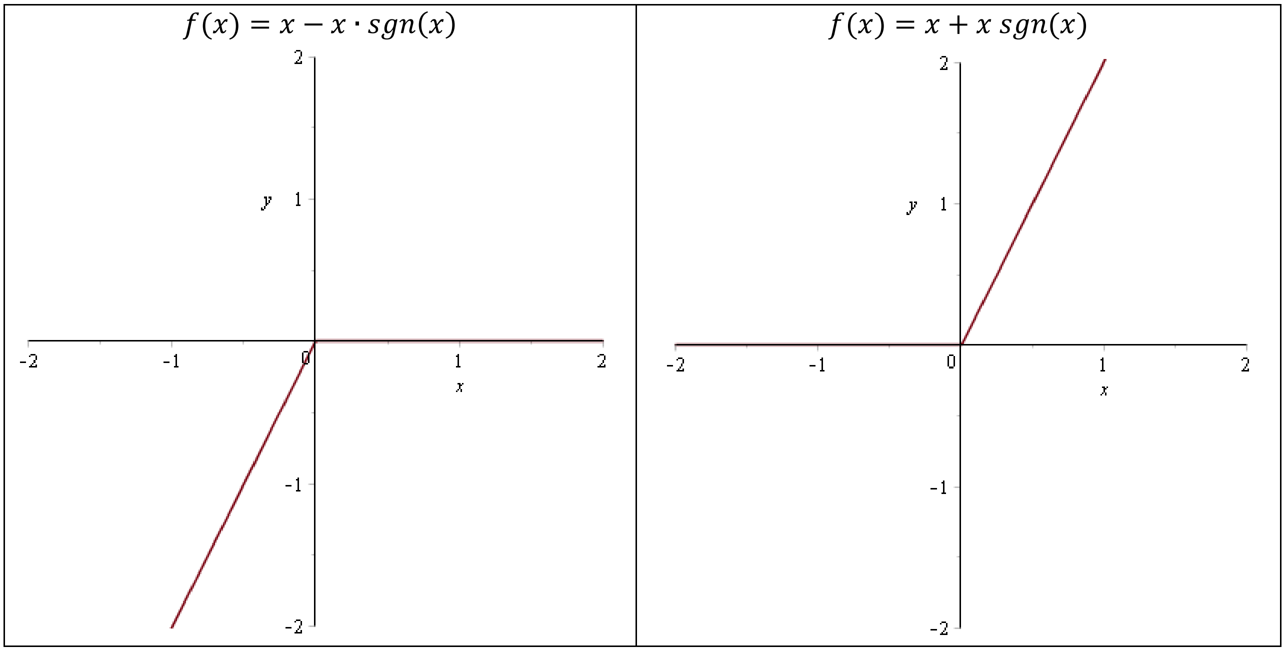
An alternative way to think about the minimum value function is by using the sign (sometimes called signum) function, $sgn(x)$. The sign function is defined as follows:

$$sgn(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

a) Graph the functions $f(x) = sgn(x)$ and $f(x) = x\, sgn(x)$



b) Graph the functions $f(x) = x - x\, sgn(x)$ and $f(x) = x + x\, sgn(x)$



c) Determine the values of a, b, c, u, v, w for the following functions

$$f(x) = \begin{cases} a, & b \le c \\ u, & v > w \end{cases}$$

i. $f(x) = x + x\, sgn(x)$

$$f(x) = \begin{cases} 0, & x \le 0 \\ 2x, & x > 0 \end{cases}$$

ii. $f(x) = x\, sgn(x)$

$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$$

iii. $f(x) = g\big(x - x\, sgn(x)\big)$

$$f(x) = \begin{cases} 2gx, & x \le 0 \\ 0, & x > 0 \end{cases}$$

iv. $f(x) = g \cdot sgn(x) + h$

$$f(x) = \begin{cases} -g + h, & x \le 0 \\ g + h, & x > 0 \end{cases}$$

v. $f(x) = gx \cdot sgn(x + h)$

$$f(x) = \begin{cases} -gx, & x \leq h \\ gx, & x > h \end{cases}$$

vi. $f(x_1, x_2) = sgn(x_1 - x_2)$

$$f(x_1, x_2) = \begin{cases} 1, & x_2 \leq x_1 \\ -1, & x_2 > x_1 \end{cases}$$

Consider the following alternate specification of the Leontief technology:

$$y \leq \min(x_1, x_2) = \begin{cases} x_1, & x_1 \leq x_2 \\ x_2, & x_1 > x_2 \end{cases}$$

d) Express the basic Leontief technology specified above using the sign function.

Notice that the function $f(x_1) = \frac{1}{2}\left(x_1 - x_1 sgn(x_1)\right) = \begin{cases} x_1 \ if \ x_1 > 0 \\ 0 \ if \ x_1 \leq 0 \end{cases}$

And also that $sgn(x_1 - x_2) = \begin{cases} 1, \ x_2 \leq x_1 \\ -1, \ x_2 > x_1 \end{cases}$

Therefore, the Leontief technology can be specified by:

$$f(x_1, x_2) = \frac{1}{2}(x_1 - x_1 sgn(x_1 - x_2) + x_2 - x_2 sgn(x_2 - x_1))$$

6. Consider the following specification of Leontief technology for firm A and the Cobb-Douglas technology for firm B:

$$Firm\ A\ (Leontief): Y_A = \{(y, x_1, x_2): y \leq \min(x_1, x_2)\,; y \geq 0, x_i \geq 0, i = 1,2\}$$

$$Firm\ B: (Cobb - Douglas): Y_B = \{(y, x_1, x_2): y \leq x_1^{\frac{1}{n}} x_2^{\frac{1}{n}}; y \geq 0, x_i \geq 0, i = 1,2\}$$

a) Let n=2.

i. Calculate the maximum output of the two firms, one using the Leontief technology and the other using the Cobb-Douglas technology, if each uses 1 unit of good x_1 and 1 unit of x_2.

Firm A produces $y_A = \min(1,1) = 1$ and Firm B produces $y_B = 1^{\frac{1}{2}}1^{\frac{1}{2}} = 1$.

ii. Calculate the output of the two firms if each uses 16 units of good x_1 and 1 unit of x_2.

Firm A produces $y_A = \min(16,1) = 1$ and Firm B produces $y_B = 16^{\frac{1}{2}}1^{\frac{1}{2}} = 4$

iii. Calculate the output of the two firms if each uses 64 units of good x_1 and 1 unit of x_2.

Firm A produces $y_A = \min(64,1) = 1$ and Firm B produces $y_B = 64^{\frac{1}{2}}1^{\frac{1}{2}} = 8$

b) Calculate question a) for n=4, n=8, n=100, n=1000.

		n=2	n=4	n=8	n=100	n=1000
$(x_1, x_2) = (1,1)$	Firm A	1	1	1	1	1
	Firm B	1	1	1	1	1
$(x_1, x_2) = (16,1)$	Firm A	1	1	1	1	1
	Firm B	4	2	1.414	1.028	1.003
$(x_1, x_2) = (64,1)$	Firm A	1	1	1	1	1
	Firm B	8	2.828	1.682	1.042	1.004

c) What do you notice about the calculations for parts a) and b)?

As n increases, the production of Firm B converges to the production of Firm A.

7. Suppose you are the CEO of a management consultancy. The company hires junior and senior analysts to produce reports for clients which contain recommendations for their businesses. Because of their different levels of experience, junior analysts make one half of the wage of senior analysts and work the same number of hours. However, due to their higher level of experience, senior analysts also take less time to produce a report. In fact, you have found that, on average, a senior analyst takes 80 hours to write a report, while a junior analyst takes 120 hours.

   At present, your firm produces a total of 40 reports per quarter and charges $15,000 per report.

   There are 2080 work hours per full time employee per year, and therefore 520 work hours per employee per quarter. Assume employees spend 75% of their time on project work (report writing, in this case) and 25% on overhead (administrative tasks which are not billable to a particular project such as meetings and training). All employees must be hired full time but multiple employees can work on the same report.

   a) What is the technical rate of substitution between junior analysts and senior analysts?

      The TRS is $\frac{120}{80} = 1.5$ junior analysts per senior analysts.

   b) If all of the reports were produced by senior analysts, how many would you need to have on staff?

      $$(40 \text{ reports per quarter}) * (80 \text{ hours per report}) = 3200 \text{ hours required per quarter}$$

      $$\frac{3200 \text{ hours required per quarter}}{(520 \text{ work hours per employee}) * 75\%} = 8.20 \text{ senior analysts required to write 40 reports}$$

      Therefore, 9 senior analysts must be hired to complete the workload.

   c) If all the reports were produced by junior analysts, how many would you need to have on staff?

      $$(40 \text{ reports per quarter}) * (120 \text{ hours per report}) = 4800 \text{ hours required per quarter}$$

      $$\frac{4800 \text{ hours required per quarter}}{(520 \text{ work hours per employee}) * 75\%} = 12.3 \text{ junior analysts required to write 40 reports}$$

      Therefore, 13 junior analysts must be hired to complete the workload.

d) Verify that the TRS is the same as that you calculated in part a) for 40 reports.

The exact TRS is: $\frac{12.3}{8.2} = 1.5$

However since employees must be hired full time, the effective technical rate of substitution is: $\frac{13}{9} = 1.44 < 1.5$

e) In what way does the requirement that all employees must be hired full time impact the effective technical rate of substitution? Hint: Consider the case where the company requires an additional three reports to be written for a total of 43 reports.

The effective TRS depends on the efficiency of the hiring practices. The workload may be slightly higher than a certain number of full time employees can handle. In these cases, the rate of substitution may be less than or greater than the TRS.

f) If junior analysts are paid $20 per hour and senior analysts are paid $40 per hour, which of a) or b) has the lower cost?

A total of 13 Junior Analysts must be hired to complete the 40 reports at $20 per hour.

$13 \text{ Junior Analysts} * \$20 \text{ per hour} * 520 \text{ hours per quarter} = \$135{,}200 \text{ wage cost}$

A total of 9 Senior Analysts must be hired to complete the 40 reports at $40 per hour

$9 \text{ Senior Analysts} * \$40 \text{ per hour} * 520 \text{ hours per quarter} = \$187{,}200 \text{ wage cost}$

g) Considering only wage costs:

i. What ratio of senior to junior analysts should you hire?

You would only hire junior analysts because your wage costs would be lower.

ii. Is this a realistic way to staff your company? Please explain.

However, this is not very realistic since you would not have any senior employees to train them!

h) What functional form will the production function take?

The production function will be linear.

i) Write out the production function. Comment on the substitution between junior and senior analysts.

$$y = \frac{(520 \text{ work hours per employee}) * 75\%}{80} x_1 + \frac{(520 \text{ work hours per employee}) * 75\%}{120} x_2$$

$$y = 4.875x_1 + 3.25x_2$$

where x_1 is the number of senior analysts and x_2 is the number of junior analysts.

This means that junior and senior analysts are perfect substitutes.

j) Determine whether the company can meet its workload with the following staffing levels:

i. 4 junior analysts and 5 senior analysts

$y = 4.875x_1 + 3.25x_2 = 4.875 \cdot 5 + 3.25 \cdot 4 = 37.375 < 40$

No, the company cannot complete its workload.

ii. 7 junior analysts and 4 senior analysts

$y = 4.875x_1 + 3.25x_2 = 4.875 \cdot 4 + 3.25 \cdot 7 = 42.25 > 40$

Yes the company can complete its workload.

iii. 3 junior analysts and 7 senior analysts

$y = 4.875x_1 + 3.25x_2 = 4.875 \cdot 7 + 3.25 \cdot 3 = 43.875 > 40$

Yes the company can complete its workload.

k) For the feasible staffing levels in part j), what is the total wage cost?

$$7 \text{ Junior Analysts} * \$20 \text{ per hour} * 520 \text{ hours per quarter} + 4 \text{ Senior Analysts} * \$40 \text{ per hour} * 520 \text{ hours per quarter} = \$72{,}800 + \$83{,}200 = \$156{,}000$$

$$3 \text{ Junior Analysts} * \$20 \text{ per hour} * 520 \text{ hours per quarter} + 7 \text{ Senior Analysts} * \$40 \text{ per hour} * 520 \text{ hours per quarter} = \$31{,}200 + \$145{,}600 = \$176{,}800$$

In addition to your wage costs, you must pay overhead for rent and support staff, each of which are related to the total number of staff members you have, irrespective of whether they are senior or junior staff members. Overhead costs are $60 per hour per employee, regardless of whether the employee is spending time on project work or on administrative tasks.

l) Calculate the total costs to your company for staff as described in parts a) and b) and those staffing levels in part j) that are feasible

Case 1: All junior analysts

13 staff members ∗ $60 per hour overhead ∗ 520 hours per quarter
= $405,600 in overhead costs + $135,200 in wage costs
= $540,800 total cost

Case 2: All senior analysts

9 staff members ∗ $60 per hour ∗ 520 hours per quarter
= $280,800 in overhead costs + $187,200 in wage costs
= $468,000 total cost

Case 3: 7 senior analysts and 4 junior analysts

11 staff members ∗ $60 per hour ∗ 520 hours per quarter
= $343,200 in overhead costs + $156,000 in wage costs
= $499,200 total cost

Case 4: 3 senior analysts and 7 junior analysts

10 staff members ∗ $60 per hour ∗ 520 hours per quarter
= $312,000 in overhead costs + $176,500 in wage costs
= $488,800 total cost

i. Does that change the ratio of senior to junior analysts that you should hire? Please explain.

   Yes. Including overhead costs, having junior analysts is more costly. Therefore, the company should hire all senior analysts.

ii. Is this a realistic way to staff your company?

   Hiring only senior analysts is not a very good way to staff your company since there is no one to train and develop into senior analysts.

m) What is the new production function?

$$y = \frac{80(x_1 + 1.5x_2)}{390} + \frac{60}{520}(x_1 + x_2)$$

n) Calculate the quarterly profit of the firm for hiring all junior analysts, all senior analysts and the feasible staffing levels in part j)

Case 1: All junior analysts

\$600,000-\$540,800=\$59,200

Case 2: All senior analysts

\$600,000-\$468,000=\$132,000

Case 3: 7 senior analysts and 4 junior analysts

\$600,000-\$499,200=\$100,800

Case 4: 3 senior analysts and 7 junior analysts

\$600,000-\$488,800=\$111,200

o) Without calculating it, what kind of solution do you expect for a profit maximization problem?

You would expect infinite profit. Since the profit is positive and the production function is linear, you have constant returns to scale.

p) Verify your intuition for part o).

$$\pi(p, w_1, w_2) = \max_{x_1, x_2} \left\{ p \cdot y - w_1 x_1 - w_2 x_2 : y = \frac{80(x_1 + 1.5x_2)}{390} + \frac{60}{520}(x_1 + x_2) \right\}$$

$$\pi(p, w_1, w_2) = \max_{x_1, x_2} \left\{ 15{,}000 \cdot \frac{80(x_1 + 1.5x_2)}{390} + \frac{60}{520}(x_1 + x_2) - w_1 x_1 - w_2 x_2 \right\}$$

Which is unbounded.

q) Why is the analysis of this business inadequate? Hint: What assumptions underlie production theory that may not be met?

Although, a production function may in fact be linear, the assumption underlying production theory is that each firm is small compared to the market. Profit cannot really increase without bound because the market is not in reality without bound.

8. Consider the following input requirement set:

$$Y = \{(y, -x_1, -x_2): y \le \ln x_1 + \ln x_2 \;; y > 0, x_i > 0, i = 1,2\}$$

a) What is the firm's profit maximization problem if its output has price p and inputs x_1 and x_2 cost w_1 and w_2, respectively?

$$\pi(p, w_1, w_2) = \max_{x_1, x_2} \{p \cdot y - w_1 x_1 - w_2 x_2 : y = \ln x_1 + \ln x_2\}$$
$$= \max_{x_1, x_2} \{p(\ln x_1 + \ln x_2) - w_1 x_1 - w_2 x_2\}$$

b) What are the first order conditions for the solution?

$$\frac{\partial L}{\partial x_1} = \frac{p}{x_1} - w_1 = 0\,;$$

$$\frac{\partial L}{\partial x_1} = \frac{p}{x_2} - w_2 = 0\,;$$

$$x_1 = \frac{p}{w_1}; x_2 = \frac{p}{w_2}$$

c) What is the firm's profit as a function of input and output prices?

$$\pi(p, w_1, w_2) = p(ln\, x_1 + ln\, x_2) - w_1 x_1 - w_2 x_2 = p\left(ln\frac{p}{w_1} + ln\frac{p}{w_2}\right) - w_1\frac{p}{w_1} - w_2\frac{p}{w_2}$$

$$= p\left(ln\frac{p}{w_1} + ln\frac{p}{w_2}\right) - 2p$$

d) Verify that the profit function satisfy the following properties:

i. Increasing in p

$$\frac{\partial \pi}{\partial p} = \ln\left(\frac{p}{w_1}\right) + \ln\left(\frac{p}{w_2}\right)$$

ii. Decreasing in w

$$\frac{\partial \pi}{\partial w_1} = \frac{-p}{w_1} < 0\,;\; \frac{\partial \pi}{\partial w_2} = \frac{-p}{w_2} < 0$$

iii. Homogeneous of degree one in (p,w)

$$\pi(tp, tw_1, tw_2) = tp\left(ln\frac{tp}{tw_1} + ln\frac{tp}{tw_2}\right) - 2tp = t\left(p\left(ln\frac{p}{w_1} + ln\frac{p}{w_2}\right) - 2p\right) = \pi(p, w_1, w_2)$$

iv. Satisfies Hotelling's Lemma

$$-x_1(p, w_1, w_2) = \frac{-p}{w_1} = \frac{\partial \pi}{\partial w_1}$$

$$-x_2(p, w_1, w_2) = \frac{-p}{w_2} = \frac{\partial \pi}{\partial w_2}$$

$$y(p, w_1, w_2) = \ln\left(\frac{p}{w_1}\right) + \ln\left(\frac{p}{w_2}\right) = \frac{\partial \pi}{\partial p}$$

9. Consider the following sets:

$$Y_a = \{(y_1, y_2) : \ y_2 \leq \ e^{-y_1} - a \ \vee \ y_1 \leq 0\}$$

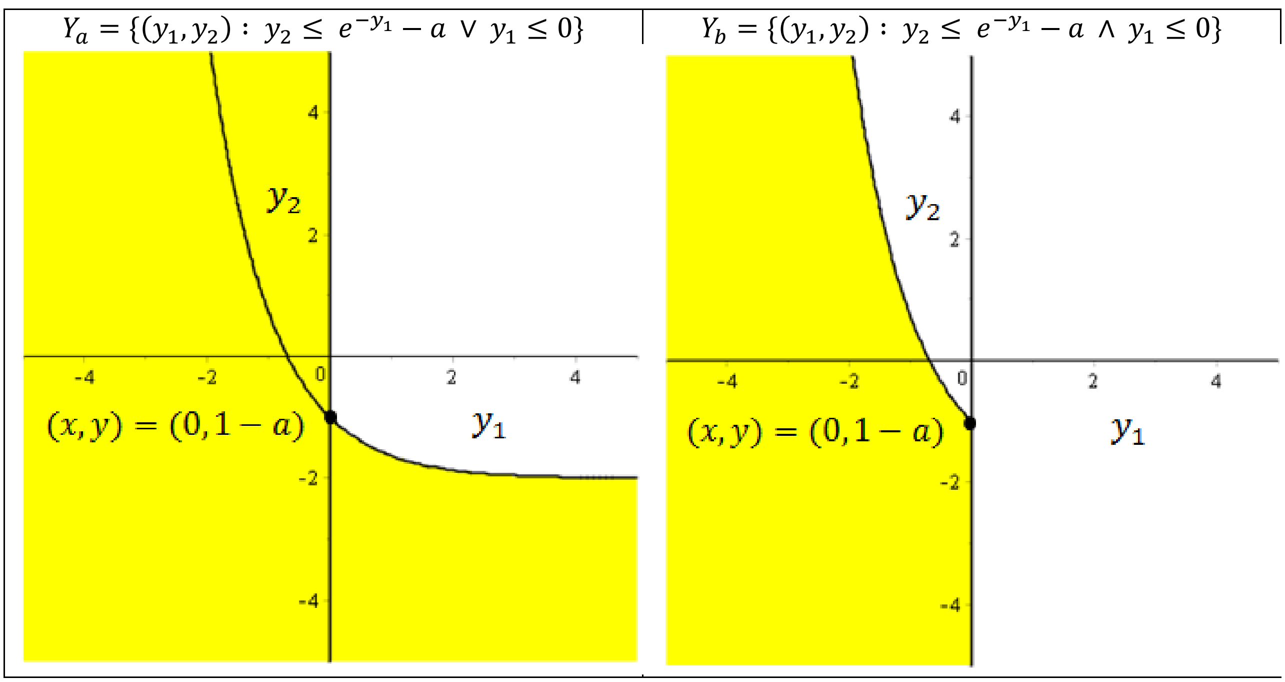
$$Y_b = \{(y_1, y_2) : \ y_2 \leq \ e^{-y_1} - a \ \wedge \ y_1 \leq 0\}$$

$$Y_c = \{(y_1, y_2) : \ y_2 \leq \ e^{-y_1} - a \ \vee \ \neg\, y_1 \leq 0\}$$

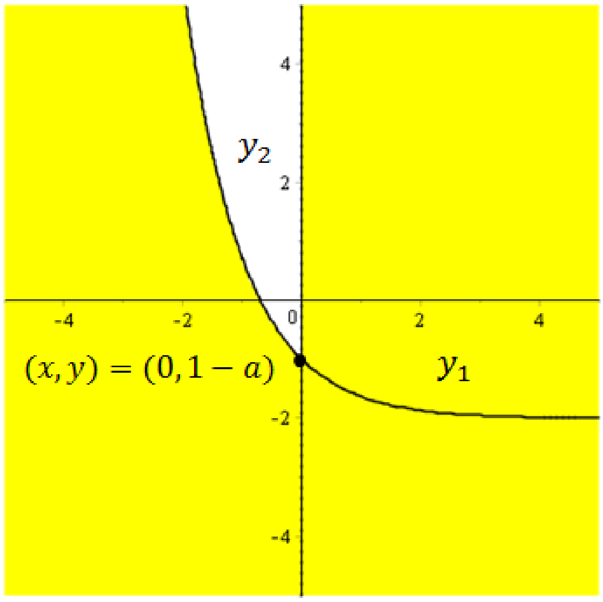
$$Y_d = \{(y_1, y_2) : \ y_2 \leq \ e^{-y_1} - a \ \wedge \ \neg y_1 \leq 0\}$$

The logical operators:	AND	OR	NOT
Are expressed by the symbols:	∧ &&	∨ ||	¬ ~ !

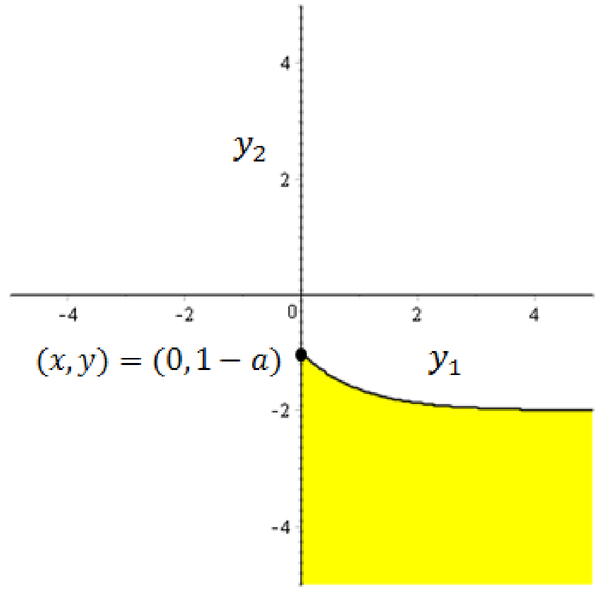
c) Graph each of the four sets.



$Y_c = \{(y_1, y_2) : y_2 \leq e^{-y_1} - a \lor \neg y_1 \leq 0\}$



$Y_d = \{(y_1, y_2) : y_2 \leq e^{-y_1} - a \land \neg y_1 \leq 0\}$



d) Are there values of a which make them legitimate production possibilities sets?

Only b can be made into a legitimate production possibilities set with the value $a = 1$

10. Consider the following input requirement set:

$$Y = \{(y, -x_1, -x_2): y \leq \ln x_1 + \ln x_2 \ ; y > 0, x_i > 0, i = 1{,}2\}$$

a) What is the firm's profit maximization problem if its output has price p and inputs x_1 and x_2 cost w_1 and w_2, respectively?

$$\max_{x_1, x_2}\{py - w_1x_1 - w_2x_2 : y \leq \ln x_1 + \ln x_2\}$$

b) What are the first order conditions for the solution?

Substituting $y = \ln x_1 + \ln x_2$ into the profit formula yields

$$\max_{x_1, x_2}\{p(\ln x_1 + \ln x_2) - w_1x_1 - w_2x_2\}$$

Therefore the first order conditions are

$$\frac{\partial L}{\partial x_1} = \frac{p}{x_1} - w_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{p}{x_2} - w_2 = 0$$

c) What is the firm's efficient output as a function of input and output prices?

From the first order conditions, we have:

$$\frac{p}{x_1} - w_1 = 0 \Leftrightarrow x_1 = \frac{p}{w_1}$$

And

$$\frac{p}{x_2} - w_2 = 0 \ \Leftrightarrow x_2 = \frac{p}{w_2}$$

Therefore,

$$y = \ln x_1 + \ln x_2 = \ln\frac{p}{w_1} + \ln\frac{p}{w_2}$$

d) What is the firm's profit as a function of input and output prices?

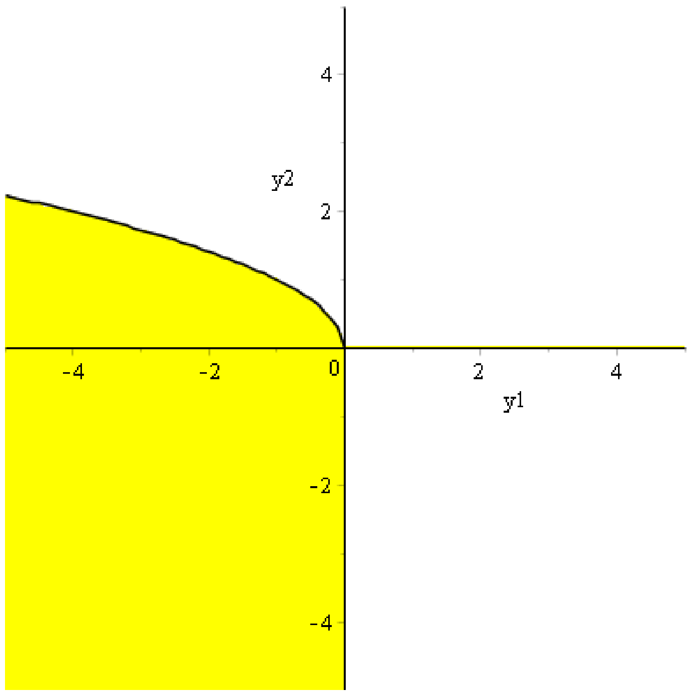
$$\pi = py - w_1x_1 - w_2x_2 = p\left(\ln\frac{p}{w_1} + \ln\frac{p}{w_2}\right) - \frac{w_1p}{w_1} - \frac{w_2p}{w_2}$$

$$= p\left(\ln\frac{p}{w_1} + \ln\frac{p}{w_2}\right) - 2p$$

11. Consider the following production possibilities set used by a firm:

$$Y = \{(y_1, y_2) : y_2 \leq \sqrt{-y_1}\}$$

a) Plot the above inequality.



b) What are the efficient points of the firm's production possibilities set?

$$\{(y_1, y_2) : y_2 = \sqrt{-y_1}\}$$

c) Show whether or not this production possibilities set is:

i. Convex

Y is convex since the line connecting any two points in Y is also in Y.

ii. Closed

Y is closed since it contains its boundary.

iii. Satisfies free disposal

Y satisfies free disposal since for any y in Y, all $y' \leq y$ is also in Y

d) Which of y_1 and y_2 is an input and which is an output?

Since $\sqrt{-y_1}$ is only real if $y_1 < 0$ we conclude that y_1 is an input.

e) What is the firm's efficient production function in terms of y, the output, and x the input?

$$y = \sqrt{x}$$

f) Write down the profit maximization problem.

$$\pi(\mathrm{p}, \mathrm{w}) = \max_{x}\{py - wx : y = \sqrt{x}\}$$

$$\pi(\mathrm{p}, \mathrm{w}) = \max_{x}\{p\sqrt{x} - wx\}$$

g) Derive the first order conditions and solve for the factor demand.

$$\frac{\partial(p\sqrt{x} - wx)}{\partial x} = \frac{1}{2}px^{-1/2} - w = 0$$

$$\frac{p}{2\sqrt{x}} = w \Leftrightarrow \sqrt{x} = \frac{p}{2w} \Leftrightarrow x = \frac{p^2}{4w^2}$$

h) What is the output in terms of p and w?

$$y = \sqrt{x} = \frac{p}{2w}$$

i) What is the profit in terms of p and w?

$$\pi(p, w) = py - wx = p\left(\frac{p}{2w}\right) - w\left(\frac{p^2}{4w^2}\right) = \frac{p^2}{2w} - \frac{p^2}{4w} = \frac{p^2}{4w}$$

j) Confirm that if the input and output prices doubled, that the profit would also double. Show generally that the profit function is homogeneous of degree 1 in p and w.

$$\pi(2p, 2w) = \frac{(2p)^2}{4 \cdot 2w} = \frac{4p^2}{8w} = \frac{p^2}{2w} = 2 \cdot \frac{p^2}{4w} = 2 \cdot \pi(p, w)$$

Suppose $(p, w) = (2,5)$.

k) What is the demand at these prices?

$$x = \frac{p^2}{4w^2} = \frac{2^2}{4 \cdot 5^2} = \frac{4}{100} = 0.04$$

l) How much is produced?

$$y = \frac{p}{2w} = \frac{2}{2 \cdot 5} = \frac{1}{5} = 0.2$$

m) What is the cost of production?

$$wx = 5 \cdot 0.04 = 0.2$$

n) How much revenue is generated?

$$py = 2 \cdot 0.2 = 0.4$$

o) What is the profit?

$$py - wx = 0.4 - 0.2 = 0.2$$

Alternatively,

$$\pi(p, w) = \frac{p^2}{4w} = \frac{2^2}{4 \cdot 5} = \frac{1}{5} = 0.2$$

Suppose that the price of the input decreases to $w = 4$ so that the price vector is $(p, w) = (2,4)$ but that the firm does not change its production.

p) Intuitively, what will happen to the firm's profit? Calculate the profit generated by the firm and verify that your intuition is correct.

The firm's profit will increase. The price of its input has decreased and the profit function is decreasing in w.

$$\pi(p, w) = py - wx = 2 \cdot 0.2 - 4 \cdot 0.04 = 0.4 - 0.16 = 0.24 > 0.2$$

q) If the firm were to adjust its production upon seeing the new prices, would the profit be higher or lower than your part (o) answer? Calculate the profit at these prices and verify that your intuition is correct.

The profit would be higher because the firm is fully optimizing.

The profit can be calculated directly using the formula derived in part i)

$$\pi(p, w) = \frac{p^2}{4w} = \frac{2^2}{4 \cdot 4} = \frac{1}{4} = 0.25 > 0.24$$

Alternatively, the profit can be calculated by determining the new production level first.

$$x = \frac{p^2}{4w^2} = \frac{2^2}{4 \cdot 4^2} = \frac{1}{16} = 0.0625$$

$$y = \sqrt{x} = \sqrt{\frac{1}{16}} = \frac{1}{4} = 0.25 \text{ or } y = \frac{p}{2w} = \frac{2}{2 \cdot 4} = \frac{1}{4} = 0.25$$

$$\pi = py - wx = 2 \cdot 0.25 - 4 \cdot 0.0625 = 0.25$$

Suppose that the price of the firm's output increases to p=3 so that the price vector is $(p, w) = (3,5)$ but that the firm does not change its production.

r) Intuitively, what will happen to the firm's profit? Calculate the profit generated by the firm and verify that your intuition is correct.

The profit will increase. The price of the output has increased and profit is increasing in p.

$$\pi(p, w) = py - wx = 3 \cdot 0.2 - 5 \cdot 0.04 = 0.6 - 0.2 = 0.4 > 0.2$$

s) If the firm were to adjust its production upon seeing the new prices, would the profit be higher or lower than your part (r) answer? Calculate the profit at these prices and verify that your intuition is correct.

The profit would be higher because the firm is optimizing.

$$\pi(p, w) = \frac{p^2}{4w} = \frac{3^2}{4 \cdot 5} = \frac{9}{20} = 0.45 > 0.4$$

Alternatively, the profit can be calculated by determining the new production level first.

$$x = \frac{p^2}{4w^2} = \frac{3^2}{4 \cdot 5^2} = \frac{9}{100} = 0.09$$

$$y = \sqrt{x} = \sqrt{\frac{9}{100}} = \frac{3}{10} = 0.3 \text{ or } y = \frac{p}{2w} = \frac{3}{2 \cdot 5} = \frac{3}{10} = 0.3$$

$$\pi = py - wx = 3 \cdot 0.3 - 5 \cdot 0.09 = 0.45$$

t) Examine the profit with and without readjustment when:

i. The input price decreases to w=1

Without readjustment

$$\pi(p, w) = py - wx = 3 \cdot 0.2 - 1 \cdot 0.04 = 0.02$$

With readjustment

$$\pi(p, w) = \frac{p^2}{4w} = \frac{3^2}{4 \cdot 1} = \frac{9}{4} = 2.25$$

ii. The output price decreases to p=1

Without readjustment

$$\pi(p, w) = py - wx = 1 \cdot 0.2 - 5 \cdot 0.04 = 0$$

With readjustment

$$\pi(p, w) = \frac{p^2}{4w} = \frac{1^2}{4 \cdot 5} = \frac{1}{20} = 0.05$$

iii. Both input and output prices simultaneously increase. Can you determine whether the profit will increase or decrease intuitively? Why or why not?

No. Profit is increasing in p and decreasing in w so the full effect is ambiguous.

u) Consider two price vectors, $(p,w) = (2,2)$ and $(p,w) = (4,6)$.

i. Calculate the optimal profit at each of these points

At $(p,w) = (2,2)$ profit is $\pi(2,2) = \frac{p^2}{4w} = \frac{2^2}{4 \cdot 2} = \frac{1}{2} = 0.5$

At $(p,w) = (4,6)$ profit is $\pi(4,6) = \frac{p^2}{4w} = \frac{4^2}{4 \cdot 6} = \frac{2}{3} \cong 0.6667$

ii. Calculate the average profit of these two price vectors.

$$\frac{\pi(2,2) + \pi(4,6)}{2} = \frac{0.5 + 0.6667}{2} = \frac{7}{12} \cong 0.5833$$

iii. Intuitively, should the profit at the average of these two prices, ie: at $(p,w) = (3,4)$ be higher or lower than your part ii) answer? Calculate the profit at (3,4) and verify your intuition is right.

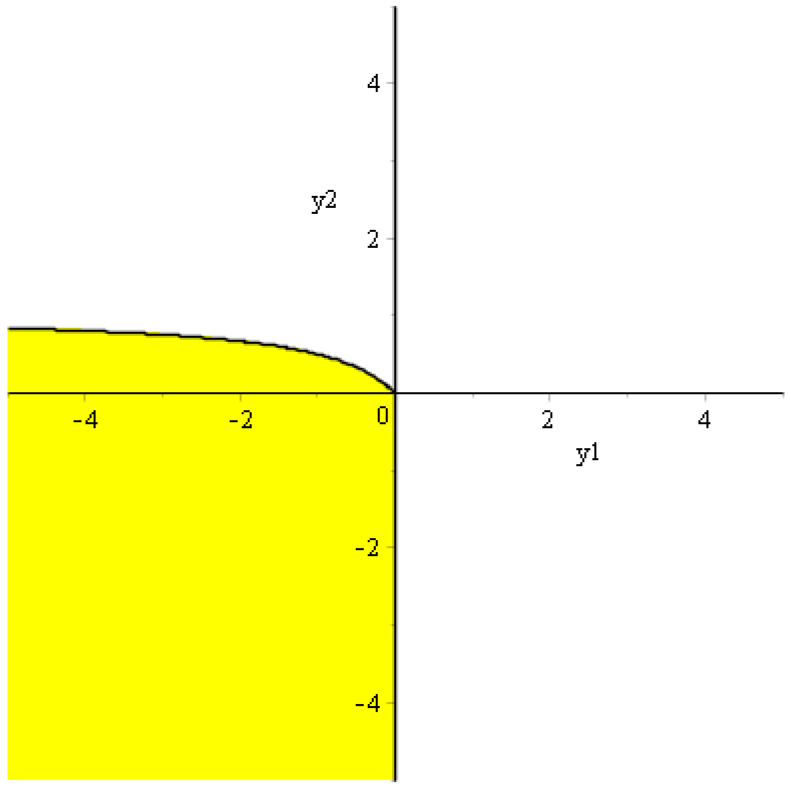
Because of convexity, the average profit should at most as high as the profit of the average of these two vectors.

$$\pi(3,4) = \frac{p^2}{4w} = \frac{3^2}{4 \cdot 4} = \frac{9}{16} = 0.5625$$

12. Consider the following production possibilities set used by a firm:

$$Y = \{(y_1, y_2) : (y_1 - 1)(y_2 - 1) \geq 1 \wedge y_1 \leq 0\}$$

a) Plot the above inequality.



b) What are the efficient points of the firm's production possibilities set?

$$\{(y_1, y_2) : (y_1 - 1)(y_2 - 1) = 1$$

c) Show whether or not this production possibilities set is:

i. Convex

Y is convex since the line connecting any two points in Y is also in Y.

ii. Closed

Y is closed since it contains its boundary.

iii. Satisfies free disposal

Y satisfies free disposal since for any y in Y, all $y' \leq y$ is also in Y.

d) Which of y_1 and y_2 is an input and which is an output?

Because of the condition, $y_1 \leq 0$, we can conclude that y_1 is an input.

e) What is the firm's efficient production function in terms of y, the output, and x the input?

$$(y_1 - 1)(y_2 - 1) = 1 \Leftrightarrow (-x - 1)(y - 1) = 1 \Leftrightarrow y - 1 = -\frac{1}{x+1} \Leftrightarrow y = 1 - \frac{1}{x+1}$$

f) Write down the profit maximization problem.

$$\pi(p, w) = \max_x \left\{ py - wx : y = 1 - \frac{1}{x+1} \right\}$$

$$\pi(p, w) = \max_x \left\{ p\left(1 - \frac{1}{x+1}\right) - wx \right\}$$

g) Derive the first order conditions and solve for the factor demand.

$$\frac{\partial\left(p\left(1 - \frac{1}{x+1}\right) - wx\right)}{\partial x} = \frac{p}{(x+1)^2} - w = 0$$

$$\frac{p}{(x+1)^2} = w \Leftrightarrow (x+1)^2 = \frac{w}{p} \Leftrightarrow x + 1 = \sqrt{\frac{p}{w}} \Leftrightarrow x = \sqrt{\frac{p}{w}} - 1$$

h) What is the output in terms of p and w?

$$y = 1 - \frac{1}{x+1} = 1 - \frac{1}{\sqrt{\frac{p}{w}} - 1 + 1} = 1 - \frac{1}{\sqrt{\frac{p}{w}}} = 1 - \sqrt{\frac{w}{p}}$$

i) What is the profit in terms of p and w?

$$\pi(p, w) = py - wx = p\left(1 - \sqrt{\frac{w}{p}}\right) - w\left(\sqrt{\frac{p}{w}} - 1\right) = p - \sqrt{pw} - \sqrt{pw} + w = p - 2\sqrt{pw} + w$$

j) Confirm that if the input and output prices doubled, that the profit would also double. Show generally that the profit function is homogeneous of degree 1 in p and w. Intuitively, what does this mean? Has the profit really doubled? Why or why not?

$$\pi(p, w) = 2p - 2\sqrt{2p \cdot 2w} + 2w = 2\left(p - 2\sqrt{pw} + w\right) = 2 \cdot \pi(p, w)$$

Although the profit has doubled, the prices have also doubled. For this reason, the relative prices of the inputs vs. the price out output have not changed.

Suppose $(p, w) = (4,4)$.

k) What is the demand at these prices, how much is produced and what is the profit?

$$x = \sqrt{\frac{p}{w}} - 1 = \sqrt{\frac{4}{4}} - 1 = 0$$

None is produced. Therefore the profit is zero.

l) What is the value of the conditional factor demand, the output and profit when p=w?

They are all zero.

Suppose (p,w)=(9,4).

m) What is the demand at these prices?

$$x = \sqrt{\frac{p}{w}} - 1 = \sqrt{\frac{9}{4}} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

n) How much is produced?

$$y = 1 - \sqrt{\frac{w}{p}} = 1 - \sqrt{\frac{4}{9}} = 1 - \frac{2}{3} = \frac{1}{3}$$

o) What is the cost of production?

$$wx = 4 \cdot \frac{1}{2} = 2$$

p) How much revenue is generated?

$$py = 9 \cdot \frac{1}{3} = 3$$

q) What is the profit?

$$\pi(9,4) = py - wx = 3 - 2 = 1$$

Or

$$\pi(9,4) = p - 2\sqrt{pw} - w = 9 - 2\sqrt{9 \cdot 4} - 4 = 9 - 2 \cdot 6 + 4 = 9 - 12 + 4 = 1$$

r) Suppose that the price of its output increases to p=16 so that the price vector is (p,w)=(16,4). However, the firm does not change its production. Intuitively, what will happen to the firm's profit? Calculate the profit generated by the firm and verify that your intuition is correct.

Intuitively, the profit will increase because the price of its output increased and the profit function is increasing in p.

$$\pi(16{,}4) = 16y - 4x = 16 \cdot \frac{1}{3} - 4 \cdot \frac{1}{2} = \frac{16}{3} - 2 = \frac{10}{3} > 1$$

s) If the firm were to adjust its production upon seeing the new prices, would the profit be higher or lower than your part (m) answer? Calculate the profit at these prices and verify that your intuition is correct.

Intuitively, the firm should be able to make more profit if it is allowed to readjust production.

$$\pi(16{,}4) = p - 2\sqrt{pw} + w = 16 - 2\sqrt{16 \cdot 4} + 4 = 16 - 2 \cdot 8 + 4 = 4 > \frac{10}{3}$$

Consumer Theory Solutions

1. What is the most general axiom necessary for the following statements to hold?

i. Any bundle is at least as good as itself.

   Reflexivity

ii. Upper contour sets do not have thick boundaries.

   Local Non-satiation

iii. At least as much of each good is at least as good.

   Weak Monotonicity

iv. A consumer prefers greater variety over less

   Strict Convexity

v. If one bundle, x_a is at least as good as another bundle, x_b, then any bundle at least as good as x_a is also at least as good as x_b.

   Transitivity

vi. A better bundle can always be found no matter how close you are to the initial bundle.

   Local Non-satiation

vii. Any two bundles can be compared and one found to be at least as good as the other.

   Completeness

viii. More of any one good is better.

   Strong monotonicity

2. Suppose the consumer's utility is given by the following utility function:

$$u = \sqrt{x_1x_2 + x_1}$$

a) What is the consumer's utility maximization problem?

$$v(\boldsymbol{p}, m) = \max_{\boldsymbol{x\epsilon X}} \left\{\sqrt{x_1x_2 + x_1} : p_1x_1 + p_2x_2 = m\right\}$$

b) What are the consumer's factor demands for goods (x_1, x_2)?

$$L = \sqrt{x_1x_2 + x_1} - \lambda(p_1x_1 + p_2x_2 - m)$$

$$\frac{\partial L}{\partial x_1} = \frac{(x_2 + 1)}{2\sqrt{x_1x_2 + x_1}} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{x_1}{2\sqrt{x_1x_2 + x_1}} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = -p_1x_1 - p_2x_2 + m = 0$$

However, note that first transforming the utility function will make the calculations easier:

$$v(\boldsymbol{p}, m) = \max_{\boldsymbol{x\epsilon X}} \{x_1x_2 + x_1 : p_1x_1 + p_2x_2 = m\}$$

$$L = x_1x_2 + x_1 - \lambda(p_1x_1 + p_2x_2 - m)$$

$$\frac{\partial L}{\partial x_1} = x_2 + 1 - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = x_1 - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = -p_1x_1 - p_2x_2 + m = 0$$

$$\frac{x_2 + 1}{x_1} = \frac{\lambda p_1}{\lambda p_2} \Leftrightarrow p_2x_2 + p_2 = x_1p_1$$

$$\Leftrightarrow m - p_1x_1 + p_2 = x_1p_1 \Leftrightarrow x_1 = \frac{m + p_2}{2p_1}$$

$$and\ p_2x_2 + p_2 = m - x_2p_2 \Leftrightarrow x_2 = \frac{m - p_2}{2p_2}$$

Note that the fact that $x_1>0$ means that $m>p_2$

c) What are the consumer's factor demands at the following price and budget levels:

i. $\boldsymbol{p} = (p_1, p_2) = (1, 4); m = 10$

$$x_1 = \frac{m + p_2}{2p_1} = \frac{10 + 4}{2 \cdot 1} = \frac{14}{2} = 7\ ;\ x_2 = \frac{m - p_2}{2p_2} = \frac{10 - 4}{2 \cdot 4} = \frac{6}{8} = \frac{3}{4}$$

ii. $\boldsymbol{p} = (p_1, p_2) = (1, 4); m = 20$

$$x_1 = \frac{m + p_2}{2p_1} = \frac{20 + 4}{2 \cdot 1} = \frac{24}{2} = 12\ ;\ x_2 = \frac{m - p_2}{2p_2} = \frac{20 - 4}{2 \cdot 4} = \frac{16}{8} = 2$$

iii. $\boldsymbol{p} = (p_1, p_2) = (1, 2); m = 10$

$$x_1 = \frac{m + p_2}{2p_1} = \frac{10 + 2}{2 \cdot 1} = \frac{12}{2} = 6\ ;\ x_2 = \frac{m - p_2}{2p_2} = \frac{10 - 2}{2 \cdot 2} = \frac{8}{4} = 2$$

iv. $\boldsymbol{p} = (p_1, p_2) = (1, 2); m = 20$

$$x_1 = \frac{m + p_2}{2p_1} = \frac{20 + 2}{2 \cdot 1} = \frac{22}{2} = 11\ ;\ x_2 = \frac{m - p_2}{2p_2} = \frac{20 - 2}{2 \cdot 2} = \frac{18}{4} = 4.5$$

d) What is the consumer's indirect utility function?

$$v(p_1, p_2, m) = \left(\frac{m + p_2}{2p_1}\right)\left(\frac{m - p_2}{2p_2}\right) + \left(\frac{m + p_2}{2p_1}\right) = \frac{(m + p_2)^2}{4p_1p_2}$$

e) Verify that the consumer's indirect utility function satisfies the following properties:

i. Non-increasing in p

$$\frac{\partial v(p_1, p_2, m)}{\partial p_1} = \frac{-(m + p_2)^2}{4p_2p_1^2} < 0$$

$$\frac{\partial v(p_1, p_2, m)}{\partial p_2} = -\frac{(m - p_2)(m + p_2)}{2\ p_1p_2^2} < 0$$

Note: The value of $(m - p_2)$ is positive since $x_1>0$

ii. Non-decreasing in m

$$\frac{\partial v(p_1, p_2, m)}{\partial m} = \frac{2(m + p_2)}{4p_1p_2} = \frac{(m + p_2)}{2p_1p_2} > 0$$

iii. Satisfies Roy's Identity

$$\frac{-\frac{\partial v(p_1,p_2,m)}{\partial p_1}}{\frac{\partial v(p_1,p_2,m)}{\partial m}} = \frac{-\frac{-(m+p_2)^2}{4p_2p_1^2}}{\frac{(m+p_2)}{2p_1p_2}} = \frac{(m+p_2)^2}{4p_2p_1^2}\cdot\frac{2p_1p_2}{(m+p_2)} = \frac{m+p_2}{p_1} = x_1$$

$$\frac{-\frac{\partial v(p_1,p_2,m)}{\partial p_2}}{\frac{\partial v(p_1,p_2,m)}{\partial m}} = \frac{\frac{(m-p_2)(m+p_2)}{2\,p_1p_2^2}}{\frac{(m+p_2)}{2p_1p_2}} = \frac{(m-p_2)(m+p_2)}{2\,p_1p_2^2}\cdot\frac{2p_1p_2}{(m+p_2)} = \frac{m-p_2}{p_2} = x_2$$

iv. Homogeneous of degree 0 in (p_1, p_2, m)

$$v(tp_1, tp_2, tm) = \frac{(tm+tp_2)^2}{4tp_1tp_2} = \frac{t^2(m+p_2)^2}{\mathrm{t}^2 4p_1p_2} = \frac{(m+p_2)^2}{4p_1p_2} = v(p_1,p_2,m)$$

f) The condition $(m - p_2) > 0$ was mentioned in passing. What happens if $p_2 > m$?

If $p_2 > m$ then the consumer does not demand any of good x_2. In this case the consumer spends his entire budget on x_1.

g) What is the consumer's expenditure function?

Using the identity $e(\boldsymbol{p}, v(\boldsymbol{p}, m)) \equiv m$ and $v(\boldsymbol{p}, v(\boldsymbol{p}, m)) \equiv u$

$$\frac{(m+p_2)^2}{4p_1p_2} = u \Leftrightarrow (m+p_2)^2 = 4up_1p_2 \Leftrightarrow m + p_2 = 2\sqrt{up_1p_2} \Leftrightarrow m = 2\sqrt{up_1p_2} - p_2$$

$$e(p_1,p_2,u) = 2\sqrt{up_1p_2} - p_2$$

h) Verify that the following expenditure function is non-decreasing in p

$$\frac{\partial e(p_1,p_2,u)}{\partial p_1} = \sqrt{\frac{p_2u}{p_1}} > 0$$

$$\frac{\partial e(p_1,p_2,u)}{\partial p_2} = \sqrt{\frac{p_1u}{p_2}} - 1 > 0 \quad \text{Since} \sqrt{\frac{p_1u}{p_2}} > 1$$

i) Verify that the following expenditure function is homogeneous of degree one in p

$$e(tp_1, tp_2, u) = 2\sqrt{utp_1tp_2} - tp_2 = t\left(2\sqrt{up_1p_2} - p_2\right) = te(p_1,p_2,u)$$

j) Derive the Hicksian Demands for the consumer using the identity:

$\boldsymbol{h}(\boldsymbol{p}, u) \equiv \boldsymbol{x}(\boldsymbol{p}, e(\boldsymbol{p}, u))$

$$h_1(p_1, p_2, u) = x_1\big(p_1, p_2, e(p_1, p_2, u)\big) = \frac{m + p_2}{2p_1} = \frac{2\sqrt{up_1p_2} - p_2 + p_2}{2p_1} = \frac{2\sqrt{up_1p_2}}{2p_1} = \sqrt{\frac{up_2}{p_1}}$$

$$h_2(p_1, p_2, u) = x_2\big(p_1, p_2, e(p_1, p_2, u)\big) = \frac{m - p_2}{2p_2} = \frac{2\sqrt{up_1p_2} - p_2 - p_2}{2p_2} = \frac{2\sqrt{up_1p_2} - 2p_2}{2p_2}$$
$$= \sqrt{\frac{up_1}{p_2}} - 1$$

k) Verify that the Hicksian demands satisfy Shephard's Lemma.

$$h_1(p_1, p_2, u) = \sqrt{\frac{up_2}{p_1}} = \frac{\partial e(p_1, p_2, u)}{\partial p_1}$$

$$h_2(p_1, p_2, u) = \sqrt{\frac{up_1}{p_2}} - 1 = \frac{\partial e(p_1, p_2, u)}{\partial p_2}$$

3. Suppose the consumer's utility is given by the following utility function:

$$u = -\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2}$$

a) What is the consumer's utility maximization problem?

$$v(\boldsymbol{p}, m) = \max_{x \epsilon X} \left\{-\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2} : p_1 x_1 + p_2 x_2 = m\right\}$$

b) Solve the consumer's utility maximization problem by setting up the Lagrangian.

$$L = -\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2} - \lambda(p_1 x_1 + p_2 x_2 - m)$$

$$\frac{\partial L}{\partial x_1} = -\frac{x_1 - 2}{\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2}} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = -\frac{x_2 - 4}{\sqrt{(x_1 - 2)^2 + (x_2 - 4)^2}} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = -p_1 x_1 - p_2 x_2 + m = 0$$

Therefore:

$$\frac{x_1 - 2}{x_2 - 4} = \frac{p_1}{p_2} \Leftrightarrow p_2 x_1 - 2p_2 = p_1 x_2 - 4p_1 \Leftrightarrow p_2 x_1 - 2p_2 = p_1\left(\frac{p_1 x_1}{p_2} - \frac{m}{p_2}\right) - 4p_1$$

$\Leftrightarrow x_1 = \frac{mp_1 - 4p_1 p_2 + 2p_2^2}{p_1^2 + p_2^2}$ and $x_2 = \frac{mp_2 + 4p_1^2 - 2p_1 p_2}{p_1^2 + p_2^2}$

c) If $\boldsymbol{p} = (p_1, p_2) = (1, 1)$ *and* $m = 8$ what are the consumer's factor demands?

$x_1 = \frac{8 \cdot 1 - 4 \cdot 1 \cdot 1 + 2 \cdot 1^2}{1^2 + 1^2} = \frac{8 - 4 + 2}{2} = \frac{6}{2} = 3$ and $x_2 = \frac{8 \cdot 1 + 4 \cdot 1^2 + 2 \cdot 1 \cdot 1}{1^2 + 1^2} = \frac{8 + 4 + 2}{2} = \frac{10}{2} = 5$

d) What is the consumer's utility at this level of demand?

$$u = -\sqrt{(3 - 2)^2 + (5 - 4)^2} = -\sqrt{1^2 + 1^2} = -\sqrt{2}$$

e) Is the bundle $\boldsymbol{x} = (x_1, x_2) = (2,4)$ affordable at $\boldsymbol{p} = (p_1, p_2) = (1, 1)$?

$$p_1 x_1 + p_2 x_2 = 1 \cdot 2 + 1 \cdot 4 = 2 + 4 = 6 < m = 8$$

f) What is the consumer's utility at the point $= (x_1, x_2) = (2,4)$?

$$u = -\sqrt{(2-2)^2 + (4-4)^2} = -\sqrt{0+0} = 0$$

g) Compare the consumer's utility from part f) and part d). What do you notice?

The utility in part f) is larger than the utility in part d), but both are affordable.

h) Why did the normal procedure fail? Hint: Consider whether all of the axioms are indeed met. Specifically examine the point $\boldsymbol{x} = (x_1, x_2) = (2,4)$

The preferences are not locally non-satiated. There is a bliss point at $\boldsymbol{x} = (x_1, x_2) = (2,4)$. This means that the budget constraint is not met with equality. Therefore, when setting up the Lagrangian, there was an error – the condition $p_1 x_1 + p_2 x_2 = m$ is not a constraint on the maximization problem. In fact, the maximization problem can be solved directly without the external budget condition.

4. Suppose the following information is collected about the behaviour of a consumer in three time periods.

	Time Period 1	Time Period 2	Time Period 3
Prices	$\boldsymbol{p}^1 = (2,2)$	$\boldsymbol{p}^2 = (1,2)$	$\boldsymbol{p}^3 = (2,3)$
Person A's Choice	$\boldsymbol{x}_A^1 = (2,2)$	$\boldsymbol{x}_A^2 = (2,1)$	$\boldsymbol{x}_A^3 = (3,2)$

a) What is Person A's expenditure in period 1?

$$\boldsymbol{p}^1 \cdot \boldsymbol{x}_A^1 = p_1^1 x_{1A}^1 + p_2^1 x_{2A}^1 = 2 \cdot 2 + 2 \cdot 2 = 8$$

b) Could have Person A afforded the bundles he consumed in periods 2 and 3?

$$\boldsymbol{p}^2 \cdot \boldsymbol{x}_A^1 = p_1^2 x_{1A}^1 + p_2^2 x_{2A}^1 = 1 \cdot 2 + 2 \cdot 2 = 6$$

Yes, the consumer could have afforded bundle 2 in period 1.

$$\boldsymbol{p}^3 \cdot \boldsymbol{x}_A^1 = p_1^3 x_{1A}^1 + p_2^3 x_{2A}^1 = 1 \cdot 2 + 3 \cdot 2 = 8$$

Yes, the consumer could have afforded bundle 3 in period 1.

c) Can you draw any conclusions on which bundle is preferred based on your a) and b) part answer?

Both bundles 1 and 2 were affordable at time period 1, but bundle 1 was chosen. Therefore we can conclude that:

$$\boldsymbol{x}_A^1 \; R^D \boldsymbol{x}_A^2 \; and \; \boldsymbol{x}_A^1 \; R^D \boldsymbol{x}_A^3$$

d) Repeat parts a) through c) for bundles 2 and 3.

Bundle 2

Expenditure: $\boldsymbol{p}^2 \cdot \boldsymbol{x}_A^2 = p_1^2 x_{1A}^2 + p_2^2 x_{2A}^2 = 1 \cdot 2 + 2 \cdot 1 = 4$

Affordability: $\boldsymbol{p}^1 \cdot \boldsymbol{x}_A^2 = p_1^1 x_{1A}^2 + p_2^1 x_{2A}^2 = 2 \cdot 2 + 2 \cdot 1 = 6$

$$\boldsymbol{p}^3 \cdot \boldsymbol{x}_A^2 = p_1^3 x_{1A}^2 + p_2^3 x_{2A}^2 = 1 \cdot 2 + 3 \cdot 1 = 5$$

The consumer could not have afforded $\boldsymbol{x}_A^2$ in periods 1 or 3. Therefore we cannot make any conclusion about the desirability of bundle $\boldsymbol{x}_A^2$.

Bundle 3

Expenditure: $\boldsymbol{p}^3 \cdot \boldsymbol{x}_A^3 = p_1^3 x_{1A}^3 + p_2^3 x_{2A}^3 = 1 \cdot 3 + 3 \cdot 2 = 9$

Affordability: $\boldsymbol{p}^1 \cdot \boldsymbol{x}_A^3 = p_1^1 x_{1A}^3 + p_2^1 x_{2A}^3 = 2 \cdot 3 + 2 \cdot 2 = 10$

$$\boldsymbol{p}^2 \cdot \boldsymbol{x}_A^3 = p_1^2 x_{1A}^3 + p_2^2 x_{2A}^3 = 1 \cdot 3 + 2 \cdot 2 = 7$$

The consumer could not have afforded $\boldsymbol{x}_A^3$ in period 3. Therefore we cannot make any conclusion about the desirability of bundle $\boldsymbol{x}_A^3$

However, the consumer could have afforded $\boldsymbol{x}_A^3$ in period 2. Therefore we can conclude:

$$\boldsymbol{x}_A^3 \, R^D \boldsymbol{x}_A^2$$

e) Do the preferences of Person A satisfy WARP?

The results from the calculation show that: $\boldsymbol{x}_A^3 \, R^D \boldsymbol{x}_A^2$ and $\boldsymbol{x}_A^1 \, R^D \boldsymbol{x}_A^2$ and $\boldsymbol{x}_A^1 \, R^D \boldsymbol{x}_A^3$

Which imply that: $\boldsymbol{x}_A^1 \, R^D \boldsymbol{x}_A^3 \, R^D \boldsymbol{x}_A^2$

The preferences satisfy WARP.

5. Suppose that a consumer is given an endowment $\omega = (\omega_1, \omega_2, \omega_3) = (2,1,3)$

This consumer's utility function is given by:

$$u(x_1, x_2, x_3) = x_1 x_2 + x_3$$

a) What is the consumer's utility with their initial endowment?

$$u(\omega_1, \omega_2, \omega_3) = u(2,1,3) = 2 \cdot 1 + 3 = 5$$

b) What is the consumer's maximization problem?

$$v(\boldsymbol{p}, m) = \max_{\boldsymbol{x \epsilon X}} \{x_1 x_2 + x_3 : p_1 x_1 + p_2 x_2 + p_3 x_3 = p_1 \omega_1 + p_2 \omega_2 + p_3 \omega_3\}$$

c) Suppose that the consumer was faced with prices $p = (p_1, p_2, p_3) = (1,1,4)$. What is the consumer's budget constraint?

$$m = p_1 \omega_1 + p_2 \omega_2 + p_3 \omega_3 = 1 \cdot 2 + 1 \cdot 1 + 4 \cdot 3 = 3 + 1 + 12 = 16$$

d) What are the consumer's demands at these prices?

$$L = x_1 x_2 + x_3 - \lambda(x_1 + x_2 + 4x_3 - 16)$$

$$\frac{\partial L}{\partial x_1} = x_2 - \lambda = 0 \Leftrightarrow x_2 = \lambda$$

$$\frac{\partial L}{\partial x_2} = x_1 - \lambda = 0 \Leftrightarrow x_1 = \lambda$$

$$\frac{\partial L}{\partial x_3} = x_3 - 4\lambda = 0 \Leftrightarrow x_3 = 4\lambda$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + 4x_3 - 16 = 0$$

Therefore:

$$x_1 = x_2 = 4x_3$$

$$x_1 + x_1 + x_1 - 16 = 0 \Leftrightarrow x_1 = \frac{16}{3}; x_2 = \frac{16}{3}$$

$$4x_3 = \frac{16}{3} \Leftrightarrow x_3 = \frac{1}{4} \cdot \frac{16}{3} = \frac{16}{12} = \frac{4}{3}$$

e) Are the demands of each good higher or lower than the endowment?

$$x_1 = \frac{16}{3} = 5.33 > 2 = \omega_1$$

$$x_2 = \frac{16}{3} = 5.33 > 1 = \omega_2$$

$$x_3 = \frac{4}{3} = 1.33 < 4 = \omega_3$$

The consumer trades away good x_3 to purchase goodsx_1 and x_2

f) Verify that the consumer's utility is higher after the exchange.

$$u(x_1, x_2, x_3) = \frac{16}{3} \cdot \frac{16}{3} + \frac{4}{3} = \frac{256}{9} + \frac{3 \cdot 4}{3 \cdot 3} = \frac{256 + 12}{9} = 29.78 > 5$$