MAT 2322 C   Assignment # 1

due Thursday, January 28th

- The assignments are due at the beggining of the lecture.
- Late assignments will not be accepted.

Find and classify the critical points:
1. $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
2. $f(x,y) = e^{x^2+y^2-4x}$

3. Find the absolute extrema of $f(x,y) = (4x - x^2)\cos y$ on the rectangular plate $1 \leq x \leq 3, \; -\pi/4 \leq y \leq \pi/4$.

4. Find the maximum and mininum values of $f(x,y) = x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

5. Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.

MAT 2322 Assignment #1 : Solutions

1. $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

so $f_x = 12x - 6x^2 + 6y = 6(2x - x^2 + y)$

and $f_y = 6y + 6x = 6(y+x)$

then $f_y = 0 \Rightarrow 6(y+x) = 0 \Rightarrow y + x = 0 \Rightarrow y = -x$

and so $f_x = 0 \Rightarrow 6(2x - x^2 + y) = 0 \Rightarrow 2x - x^2 + y = 0$

plug in $y = -x$, we get $2x - x^2 - x = 0$ or $x - x^2 = 0$

so $x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$

and so the critical points are $(0,0)$ and $(1,-1)$

$$\left.\begin{aligned} f_{xx} &= 12 - 12x \\ f_{yy} &= 6 \\ f_{xy} &= 6 \end{aligned}\right\} \text{ so } D = f_{xx} f_{yy} - (f_{xy})^2 = (12-12x)(6) - (6)^2 = 36 - 72x = 36(1-2x)$$

then $D(0,0) = 36 > 0$, $f_{xx}(0,0) = 12 > 0$ $\boxed{\therefore (0,0) \text{ is a local min}}$

and $D(1,-1) = -36 < 0$ $\boxed{\therefore (1,-1) \text{ is a saddle point}}$

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2. $f(x,y) = e^{x^2+y^2-4x}$

so $f_x = (2x-4)e^{x^2+y^2-4x}$

and $f_y = 2y\, e^{x^2+y^2-4x}$

$$\left.\begin{aligned} f_x = 0 &\Rightarrow 2x - 4 = 0 \Rightarrow x = 2 \\ f_y = 0 &\Rightarrow 2y = 0 \Rightarrow y = 0 \end{aligned}\right\} \text{ so } (2,0) \text{ is the only critical point}$$

$f_{xx} = 2e^{x^2+y^2-4x} + (2x-4)^2 e^{x^2+y^2-4x}$

$= (4x^2 - 16x + 18)e^{x^2+y^2-4x}$

$f_{yy} = 2e^{x^2+y^2+4x} + (2y)^2 e^{x^2+y^2-4x} = (4y^2+2)e^{x^2+y^2-4x}$

$f_{xy} = 2y(2x-4)e^{x^2+y^2-4x}$

$D = \left[(4y^2+2)(4x^2-16x+18) - 4y^2(2x-4)^2\right] e^{2(x^2+y^2-4x)}$

$D(2,0) > 0$, $f_{xx}(2,0) > 0$ $\boxed{\therefore (2,0) \text{ is a local min}}$

3. $f(x,y) = (4x - x^2)\cos y$ on $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$

look for critical points in the interior:

$f_x = (4-2x)\cos y = 2(2-x)\cos y$

$f_y = -(4x-x^2)\sin y$

then $f_x = 0 \Rightarrow 2(2-x)\cos y = 0 \Rightarrow x = 2$ or $\cos y = 0$

but $\cos y > 0$ on $-\pi/4 \le y \le \pi/4 \Rightarrow x = 2$ only

and $f_y = 0 \Rightarrow -(4x-x^2)\sin y = 0$

since $x = 2$, $4x - x^2 \neq 0$, so must have $\sin y = 0$

and hence $y = 0$

so there is one critical point at $(2,0)$, which is in the interior of the region, $f(2,0) = (4(2)-(2)^2)\cos(0) = 4$

along the boundary:

if $x = 1$, then $f(1,y) = 3\cos y$

if $x = 3$, then $f(3,y) = 3\cos y$

so we look for max and min of $g(y) = 3\cos y$ on $[-\pi/4, \pi/4]$

$g(-\pi/4) = g(\pi/4) = 3/\sqrt{2}$

$g'(y) = -3\sin y = 0$ if $y = 0$ $\quad g(0) = 3$

if $y = -\pi/4$, then $f(x,-\pi/4) = (4x-x^2)/\sqrt{2}$

if $y = \pi/4$, then $f(x,\pi/4) = (4x-x^2)/\sqrt{2}$

so we look for max and min of $h(x) = (4x-x^2)/\sqrt{2}$ on $[1,3]$

$h(1) = h(3) = 3/\sqrt{2}$

$h'(x) = (4-2x)/\sqrt{2} = 0$ if $x = 2$, $h(2) = 4/\sqrt{2} = 2\sqrt{2}$

$\therefore$ abs max is 4 at $(2,0)$

abs min is $3/\sqrt{2}$ at $(1,-\pi/4)$, $(1,\pi/4)$, $(3,-\pi/4)$, $(3,\pi/4)$

4. $f(x,y) = x^2+y^2$ subject to $g(x,y) = x^2-2x+y^2-4y=0$

$\nabla f = \lambda \nabla g \Rightarrow f_x = \lambda g_x \Rightarrow 2x = \lambda(2x-2)$

$f_y = \lambda g_y \Rightarrow 2y = \lambda(2y-4)$

or $x = \lambda(x-1)$ $\quad x=0 \Rightarrow \lambda=0$

$y = \lambda(y-2)$ $\quad y=0 \Rightarrow \lambda = 0$

if $x=1$, $x=0$ ※ $\}$ so $x\neq 1, y\neq 2$

if $y=2$, $y=0$ ※

thus $\lambda = \frac{x}{x-1}$ and $\lambda = \frac{y}{y-2}$

so $\frac{x}{x-1} = \frac{y}{y-2} \Rightarrow xy-2x = xy-y \Rightarrow y=2x$

plug into $g(x,y)$: $x^2-2x+(2x)^2-4(2x)=0$

or $x^2-2x+4x^2-8x = 5x^2-10x = 5x(x-2)=0$

then $x=0 \Rightarrow (0,0)$ or $x=2 \Rightarrow (2,4)$

$f(0,0) = 0$

$f(2,4) = (2)^2+(4)^2 = 20$

$\therefore$ $\boxed{\text{the max is 20 at (2,4) and the min 0 at (0,0)}}$

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5. the distance of point (x,y,z) to the origin is

$d(x,y,z) = \sqrt{x^2+y^2+z^2}$

but the point that minimizes this distance will also minimize its square $d^2(x,y,z) = x^2+y^2+z^2$ (which is much easier to work with)

so we minimize $x^2+y^2+z^2 = f(x,y,z)$ subject to the constraints

$g_1(x,y,z) = y+2z=12$ and $g_2(x,y,z) = x+y=6$

then $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ will give us

$$f_x = \lambda_1 g_{1x} + \lambda_2 g_{2x} \Rightarrow 2x = \lambda_2$$
$$f_y = \lambda_1 g_{1y} + \lambda_2 g_{2y} \Rightarrow 2y = \lambda_1 + \lambda_2$$
$$f_z = \lambda_1 g_{1z} + \lambda_2 g_{2z} \Rightarrow 2z = 2\lambda_1$$

so we have that $\lambda_2 = 2x$ and $\lambda_1 = z$

so $2y = \lambda_1 + \lambda_2$ becomes $2y = z + 2x$ or $2x - 2y + z = 0$

then we solve
$$2x - 2y + z = 0 \quad (1)$$
$$y + 2z = 12 \quad (2)$$
$$x + y = 6 \quad (3)$$

$(1) - 2\times(3) \Rightarrow -4y + z = -12 \quad (4)$

then $4\times(2) + (4) \Rightarrow 9z = 36 \Rightarrow z = 4 \Rightarrow y = 4 \Rightarrow x = 2$

$\therefore$ $\boxed{\text{the point is } (2,4,4)}$ (this point will give the min as there is no max in this situation)