

Due Wednesday, February 3rd, by 2 pm in the Economics Office, SS 454. The office is closed during lunch hour, noon – 1 pm.

Students may work in small groups of 2 students on assignments, and a group file is to be handed in. Students in the group will receive the same mark for the assignment. Please remember to do a cover sheet with your name(s), the course number and your section, number all the pages, clearly label which question and part you are answering, and staple your assignment.

Your assignment should be done in a professional manner. Part marks will be deducted for leaving out formula, not showing calculations and all the steps, not writing sentence answers to all questions, not clearly labelling the question and the parts of the questions, not including a cover page with your name(s), not stapling the assignment, and not numbering the pages.

**The assignment will be graded in the following manner:
20% for attempting all the questions, NO PART MARKS
20% for each of questions 3, 5, 7, and 10.**

1. Express each of the following sums in summation notation and then compute where possible. Let X take the values $x_1 = -2, x_2 = -1, x_3 = 0, x_4 = 1, x_5 = 2$ and Y take the values $y_1 = -1, y_2 = -0.5, y_3 = 0.25, y_4 = 1, y_5 = 1.5$.

a) $x_1 + x_2 + x_3 + x_4 + x_5 = \sum_{i=1}^5 x_i = 0$

b) $y_3 + y_4 + y_5 = \sum_{i=3}^5 y_i = 2.75$

c) $(x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3) + (x_4 - y_4) + (x_5 - y_5) = \sum_{i=1}^5 (x_i - y_i) = \sum_{i=1}^5 x_i - \sum_{i=1}^5 y_i = -1.25$

d) $2(x_1 * y_2) + 2(x_2 * y_3) + 2(x_3 * y_4) = \sum_{i=1}^3 2(x_i * y_{i+1}) = 2 \sum_{i=1}^3 (x_i * y_{i+1}) = 1.5$

e) $(x_2/y_1) + (x_3/y_2) + (x_4/y_3) = \sum_{i=2}^4 \frac{x_i}{y_{i-1}} = 1 + 0 + 4 = 5$

f) $(x_1^2 * 2y_3) + (x_2^2 * 2y_4) + (x_3^2 * 2y_5) = \sum_{i=1}^3 (x_i^2 * 2y_{i+2}) = 2 \sum_{i=1}^3 (x_i^2 * y_{i+2}) = 4$

2. Show that $\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}$

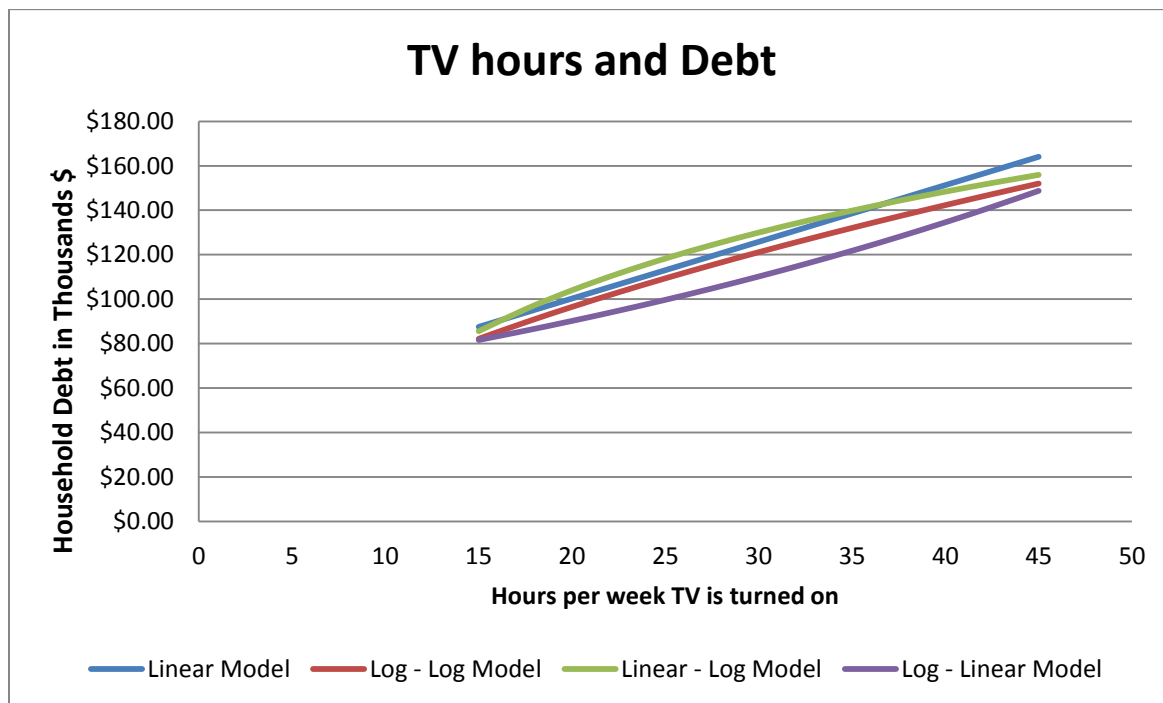
$$\begin{aligned} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1}^n (x_i y_i - \bar{x} y_i - x_i \bar{y} + \bar{x} \bar{y}) \\ &= \sum_{i=1}^n x_i y_i - \sum_{i=1}^n \bar{x} y_i - \sum_{i=1}^n x_i \bar{y} + \sum_{i=1}^n \bar{x} \bar{y} \\ &= \sum_{i=1}^n x_i y_i - \bar{x} \frac{n}{n} \sum_{i=1}^n y_i - \bar{y} \frac{n}{n} \sum_{i=1}^n x_i + n\bar{x}\bar{y} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^n x_i y_i - n\bar{x} \frac{\sum_{i=1}^n y_i}{n} - n\bar{y} \frac{\sum_{i=1}^n x_i}{n} + n\bar{x}\bar{y} \\
 &= \sum_{i=1}^n x_i y_i - n\bar{x}\bar{y} - n\bar{y}\bar{x} + n\bar{x}\bar{y} = \sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}
 \end{aligned}$$

3. An economist is interested in the possible relationship between the amount of time people are watching television and the size of their debt. A sociologist theorized that people who frequently watch television are exposed to more commercials and this in turn leads to more consumption and a higher debt. Below are four equations that were estimated by the economist with some data where x is the number of hours that a television is turned on each week and y is the total debt in the household.

$$\begin{aligned}
 y &= 49,287.23 + 2,550.53x \\
 \ln(y) &= 9.80 + 0.56 \ln(x) \\
 y &= -87,758 + 64,014.70 \ln(x) \\
 \ln(y) &= 11.01 + 0.02x
 \end{aligned}$$

- a) Plot each of the functions for $x = 15$ to $x = 45$.



- b) Calculate the slope of each function at $x = 30$. State the interpretation of the slope.
 $y = 49,287.23 + 2,550.53x$, and the slope is 2,550.53, constant, at all values of x . For each extra hour that the television is turned on there is an average increase in household debt of \$2,550.53.

$\ln(y) = 9.80 + 0.56 \ln(x)$. The slope is $0.56 \cdot (y/x)$, so for $x = 30$, $0.56 \cdot (121,136.17/30) = \$2,261.21$. A 1% increase in the hours per week the TV is turned on leads to a 0.56% increase in average household debt. When the TV is turned on for 30 hours in the week, an increase of 1 hour will be associated with an average household debt increase of \$2,261.21.

$y = -87,758 + 64,014.70 \ln(x)$. The slope is $64,014.7 \cdot (1/x)$, so for $x = 30$, $64,014.7 \cdot (1/30) = \$2,133.82$. When the TV is turned on for 30 hours in the week, an increase of 1 hour will be associated with an average household debt increase of \$2,133.82.

$\ln(y) = 11.01 + 0.02x$. The slope is $0.02 \cdot y$, so for $x = 30$, $0.02 \cdot 110,194.25 = \$2,203.89$. When the TV is turned on for 30 hours in the week, an increase of 1 hour will be associated with an average household debt increase of \$2,203.89.

- c) Calculate the elasticity of each function at $x = 30$ and give its interpretation.

$y = 49,287.23 + 2,550.53x$. The elasticity is $2,550.53 \cdot (x/y) = 2,550.53 \cdot (30/125,803.13) = 0.63$, inelastic. When the TV is turned on for 30 hours in the week, household debt increases by 0.63% when the number of hours the TV is on increases by 1%.

$\ln(y) = 9.80 + 0.56 \ln(x)$. This is the constant elasticity model, where a 1% increase in the number of hours the TV is turned on is associated with a 0.56% increase in the household debt at every value of the number of hours the TV is on. This is also inelastic.

$y = -87,758 + 64,014.70 \ln(x)$. The elasticity is $64,014.7 \cdot (1/y)$, so for $x = 30$, $64,014.7 \cdot (1/129,968.63) = 0.61$. When the TV is turned on for 30 hours in the week, an increase of 1% will be associated with an average household debt increase of 0.61%. This is also inelastic.

$\ln(y) = 11.01 + 0.02x$. The elasticity is $0.02 \cdot x$, so for $x = 30$, $0.02 \cdot 30 = 0.60$. When the TV is turned on for 30 hours in the week, an increase of 1% will be associated with an average household debt increase of 0.6%. This is also inelastic.

4. Let X be a discrete random variable that is the value shown on a single roll of a fair die.

- a) What is the probability the $X = 3$ or $X = 4$ or $X = 5$?

$$f(X = 3 \text{ or } X = 4 \text{ or } X = 5) = f(X = 3) + f(X = 4) + f(X = 5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

The probability the $X = 3$ or $X = 4$ or $X = 5$ is $\frac{1}{2}$ or **0.5**.

- b) What is the expected value of X ? What is the expected value of X^2 ?

$$E(X) = \sum_{i=1}^6 x_i f(x_i) = \left(1 \cdot \frac{1}{6}\right) + \left(2 \cdot \frac{1}{6}\right) + \left(3 \cdot \frac{1}{6}\right) + \left(4 \cdot \frac{1}{6}\right) + \left(5 \cdot \frac{1}{6}\right) + \left(6 \cdot \frac{1}{6}\right) = \frac{21}{6} = 3.5$$

The expected value of X is **3.5**.

$$E(X^2) = \sum_{i=1}^6 x_i^2 f(x_i) = \left(1^2 * \frac{1}{6}\right) + \left(2^2 * \frac{1}{6}\right) + \left(3^2 * \frac{1}{6}\right) + \left(4^2 * \frac{1}{6}\right) + \left(5^2 * \frac{1}{6}\right) + \left(6^2 * \frac{1}{6}\right)$$

$$= \frac{91}{6} = 15.167$$

The expected value of X^2 is 15.167.

- c) Find the variance of X .

$$var(X) = \sigma^2 = E(X - \mu)^2 = E(X^2) - \mu^2 = 15.167 - (3.5^2) = 2.9167.$$

The variance of X is 2.92.

- d) Find the expected value and variance of W if $W = g(X) = 4X - 2$.

$$E(W) = E(4X - 2) = 4E(X) - 2 = 4 * 3.5 - 2 = 12$$

The expected value of W is 12.

$$var(W) = var(4X - 2) = 4^2 var(X) = 16 * 2.9167 = 46.667.$$

The variance of W is 46.67.

- e) Now, assume the die has been weighted so that $P(X=1) = 1/6$, $P(X=2) = 1/3$, and the $P(X=3) = P(X=4) = P(X=5) = P(X=6) = 1/8$. Find the answers to part b) and c) under the new assumption.

$$E(X) = \sum_{i=1}^6 x_i f(x_i) = \left(1 * \frac{1}{6}\right) + \left(2 * \frac{1}{3}\right) + \left(3 * \frac{1}{8}\right) + \left(4 * \frac{1}{8}\right) + \left(5 * \frac{1}{8}\right) + \left(6 * \frac{1}{8}\right) = \frac{37}{12}$$

$$= 3.083$$

The expected value of X is 3.1.

$$E(X^2) = \sum_{i=1}^6 x_i^2 f(x_i) = \left(1^2 * \frac{1}{6}\right) + \left(2^2 * \frac{1}{3}\right) + \left(3^2 * \frac{1}{8}\right) + \left(4^2 * \frac{1}{8}\right) + \left(5^2 * \frac{1}{8}\right) + \left(6^2 * \frac{1}{8}\right)$$

$$= 12.25$$

The expected value of X^2 is 12.25.

$$var(X) = \sigma^2 = E(X - \mu)^2 = E(X^2) - \mu^2 = 12.25 - (3.083^2) = 2.743.$$

The variance of X is 2.74.

5. After watching a number of children playing games in a video arcade, a statistics practitioner estimated the following probability distribution of X , the number of games per visit.

x	1	2	3	4	5	6	7
f(x)	.05	.05	.15	.25	.30	.15	.05

- a) What is the probability that a child will play at least four games?

$$P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) = 0.25 + 0.30 + 0.15 + 0.05 = 0.75$$

The probability that a child will play at least four games is 0.75

- b) What is the mean and variance of the number of games played?

$$E(X) = \sum_{i=1}^7 x_i f(x_i)$$

$$= (1 * 0.05) + (2 * 0.05) + (3 * 0.15) + (4 * 0.25) + (5 * 0.3) + (6 * 0.15) + (7 * 0.05) = 0.05 + 0.1 + 0.45 + 1 + 1.5 + 0.9 + 0.35 = 4.35$$

$$E(X^2) = \sum_{i=1}^7 x_i^2 f(x_i)$$

$$= (1^2 * 0.05) + (2^2 * 0.05) + (3^2 * 0.15) + (4^2 * 0.25) + (5^2 * 0.3) + (6^2 * 0.15) + (7^2 * 0.05)$$

$$= 0.05 + 0.2 + 1.35 + 4 + 7.5 + 5.4 + 2.45 = 20.95$$

$$var(X) = \sigma^2 = E(X - \mu)^2 = E(X^2) - \mu^2 = 20.95 - 4.35^2 = 2.0275 \approx 2.03$$

The mean of the number of games played is 4.35 games and the variance is 2.03 games squared.

- c) Suppose that each game costs \$2.25. Use the laws of expected value and variance to determine the expected value and variance of the amount of money the arcade takes in per child.

Let R be the revenue the arcade receives for each child. $R = \$2.25 * X$

$$E(R) = E(2.25X) = 2.25E(X) = 2.25 * 4.35 = \$9.79$$

$$var(R) = var(2.25X) = 2.25^2 var(X) = 5.0625 * 2.0275 = 10.26.$$

The expected value of the revenue is \$9.79 and the variance is 10.26 dollars squared.

- d) Determine the probability distribution of the amount of money the arcade takes in per child.

R	2.25	4.5	6.75	9	11.25	13.5	15.75
f(R)	.05	.05	.15	.25	.30	.15	.05

- e) Use the probability distribution to calculate the mean and variance of the amount of money the arcade takes in per child.

$$E(R) = \sum_{i=1}^7 r_i f(r_i)$$

$$= (2.25 * 0.05) + (4.5 * 0.05) + (6.75 * 0.15) + (9 * 0.25) + (11.25 * 0.3) + (13.5 * 0.15) + (15.75 * 0.05) = 9.7875$$

$$E(R^2) = \sum_{i=1}^7 r_i^2 f(r_i)$$

$$= (2.25^2 * 0.05) + (4.5^2 * 0.05) + (6.75^2 * 0.15) + (9^2 * 0.25) + (11.25^2 * 0.3) + (13.5^2 * 0.15) + (15.75^2 * 0.05) = 106.0594$$

$$var(R) = \sigma^2 = E(R - \mu)^2 = E(R^2) - \mu^2 = 106.06 - 9.79^2 = 10.26$$

The expected value of the revenue is \$9.79 and the variance is 10.26 dollars squared.

- f) Are your answers in part c & part e identical? **YES.**
6. To gauge the relationship between education and unemployment an economist turned to the US Census, from which the following table was produced:

Education	Employed	Unemployed
Not a high school graduate	0.0975	0.0080
High school graduate	0.3108	0.0128
Some college, no degree	0.1785	0.0062
Associate's degree	0.0849	0.0023
Bachelor's degree	0.1959	0.0041
Advanced degree	0.0975	0.0015

- a) What is the probability that a high school graduate is unemployed?
 $P(\text{unemployed} \mid \text{high school graduate}) =$

$$\frac{P(\text{unemployed and high school graduate})}{P(\text{high school graduate})} = \frac{.0128}{.3108 + .0128} = \frac{.0128}{.3236} = 0.0396$$

The probability that a high school graduate is unemployed is 0.0396.

- b) Determine the probability that a randomly selected individual is employed.
 $P(\text{employed}) = .0975 + .3108 + .1785 + .0849 + .1959 + .0975 = 0.9651$

The probability that a randomly selected individual is employed is 0.9651.

- c) Find the probability that an unemployed person possesses an advanced degree.
 $P(\text{advanced degree} \mid \text{unemployed}) =$

$$\frac{P(\text{advanced degree and unemployed})}{P(\text{unemployed})} = \frac{.0015}{.0080 + .0128 + .0062 + .0023 + .0041 + .0015} = \frac{.0015}{.0349} = .0430$$

The probability that an unemployed person possesses an advanced degree is 0.0430.

- d) What is the probability that a randomly selected person did not finish high school?
 $P(\text{not a high school graduate}) = .0975 + .0080 = 0.1055$

The probability that a randomly selected person did not finish high school is 0.1055.

Data from the Office on Smoking and Health, Centers for Disease Control and Prevention, indicate that 40% of adults who did not finish high school, 34% of high school graduates, 24% of adults who completed some college (no degree), and 14% of graduates (Degree holders) smoke.

- e) Suppose that one individual is selected at random and it is discovered that the individual smokes. What is the probability that the individual is a graduate? (Use the probabilities from the table above to solve this question)

HINT:

Conditional probability: $P(A|B) = P(A \text{ and } B)/P(B)$

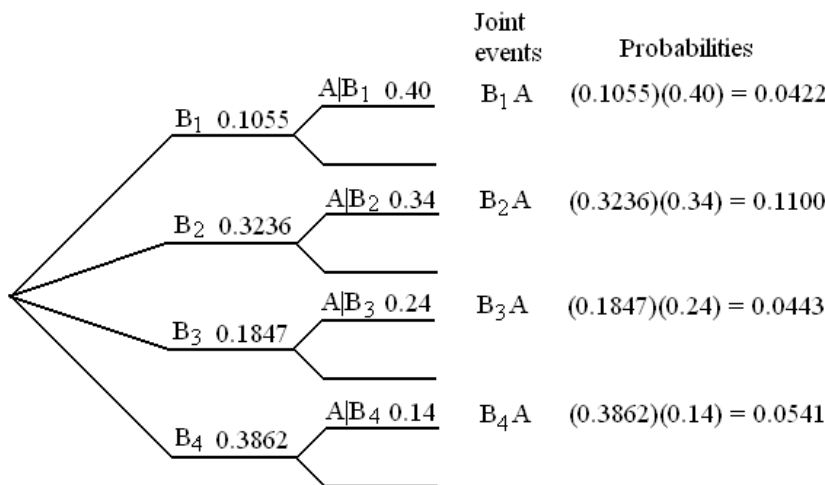
Bayes' Law Formula

$$P(A_i | B) = \frac{P(A_i)P(B | A_i)}{P(A_1)P(B | A_1) + P(A_2)P(B | A_2) + \dots + P(A_k)P(B | A_k)}$$

Define events: A = smoke, B₁ = did not finish high school, B₂ = high school graduate, B₃ = some college, no degree, B₄ = completed a degree

$P(A | B_1) = .40$, $P(A | B_2) = .34$, $P(A | B_3) = .24$, $P(A | B_4) = .14$

From Exercise above: $P(B_1) = .1055$, $P(B_2) = .3236$, $P(B_3) = .1847$, $P(B_4) = .3862$



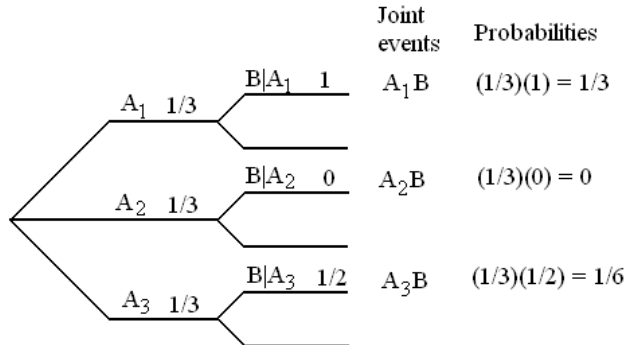
$$P(A) = .0422 + .1100 + .0443 + .0541 = 0.2506$$

$$P(B_4 | A) = .0541/.2506 = 0.2159$$

The probability that the individual is a graduate, given that the individual smokes, is 0.2159.

7. A professor and his wife are planning to take a 2-week vacation in Hawaii, but they can't decide whether to spend 1 week on each of the islands of Maui and Oahu or 2 weeks on Maui or 2 weeks on Oahu. Placing their faith in random chance they make 3 envelopes, one envelope with a brochure for Maui and a brochure for Oahu, one envelope with 2 Maui brochures and one envelope with 2 Oahu brochures. The professor's wife selects a closed envelope randomly and the professor opens the envelope and removes one brochure to find it is for Maui. What is the probability that the other brochure in the envelope is a Maui brochure?

Define the events: A_1 = envelope containing two Maui brochures is selected, A_2 = envelope containing two Oahu brochures is selected, A_3 = envelope containing one Maui and one Oahu brochures is selected. B = a Maui brochure is removed from the selected envelope.



$$P(B) = 1/3 + 0 + 1/6 = 1/2$$

$$P(A_1 | B) = \frac{P(A_1 \text{ and } B)}{P(B)} = \frac{1/3}{1/2} = 2/3$$

The probability that the other brochure in the envelope is a Maui brochure is 2/3.

8. An investor holds a portfolio consisting of two stocks. He puts 30% of his money in Stock C and 70% into stock F. Stock C has an expected return of $R_C = 10\%$ and a standard deviation of $\sigma_C = 10\%$. Stock F has an expected return of $R_F = 15\%$ and a standard deviation of $\sigma_F = 20\%$. The portfolio return is $P = 0.3 R_C + 0.7 R_F$.

- a) What is the expected return on the portfolio?

$$P = 0.3 R_C + 0.7 R_F \quad E(P) = E(0.3 R_C + 0.7 R_F) = 0.3 E(R_C) + 0.7 E(R_F)$$

$$= (0.3 * 10\%) + (0.7 * 15\%) = 13.5\%$$

The expected return on the portfolio is 13.5%.

- b) Compute the standard deviation of the portfolio if the two stocks' returns are perfectly positively correlated.

$$var(P) = 0.3^2 var(R_C) + 0.7^2 var(R_F) + 2(0.3)(0.7)cov(R_C, R_F)$$

$$\rho = 1 = \frac{cov(R_C, R_F)}{\sigma_{R_C} * \sigma_{R_F}}, \quad cov(R_C, R_F) = \sigma_{R_C} * \sigma_{R_F} = 10 * 20 = 200$$

$$var(P) = 0.3^2 var(R_C) + 0.7^2 var(R_F) + 2(0.3)(0.7)cov(R_C, R_F)$$

$$= (0.3^2 * 10^2) + (0.7^2 * 20^2) + 2(0.3)(0.7)200 = 289$$

$$\sigma_P = \sqrt{289} = 17$$

The standard deviation of the portfolio is 17%.

- c) Compute the standard deviation of the portfolio if the two stocks' returns have a correlation of 0.25.

$$var(P) = 0.3^2 var(R_C) + 0.7^2 var(R_F) + 2(0.3)(0.7)cov(R_C, R_F)$$

$$\rho = 0.25 = \frac{cov(R_C, R_F)}{\sigma_{R_C} * \sigma_{R_F}}, \quad cov(R_C, R_F) = \sigma_{R_C} * \sigma_{R_F} * .25 = 10 * 20 * .25 = 50$$

$$\begin{aligned} \text{var}(P) &= 0.3^2 \text{var}(R_C) + 0.7^2 \text{var}(R_F) + 2(0.3)(0.7) \text{cov}(R_C, R_F) \\ &= (0.3^2 * 10^2) + (0.7^2 * 20^2) + 2(0.3)(0.7)50 = 226 \end{aligned}$$

$$\sigma_P = \sqrt{226} = 15.033$$

The standard deviation of the portfolio is 15.03%.

- d) Compute the standard deviation of the portfolio if the two stocks' returns are uncorrelated.

$$\begin{aligned} \text{var}(P) &= 0.3^2 \text{var}(R_C) + 0.7^2 \text{var}(R_F) + 2(0.3)(0.7) \text{cov}(R_C, R_F) \\ &= (0.3^2 * 10^2) + (0.7^2 * 20^2) + 2(0.3)(0.7)0 = 205 \end{aligned}$$

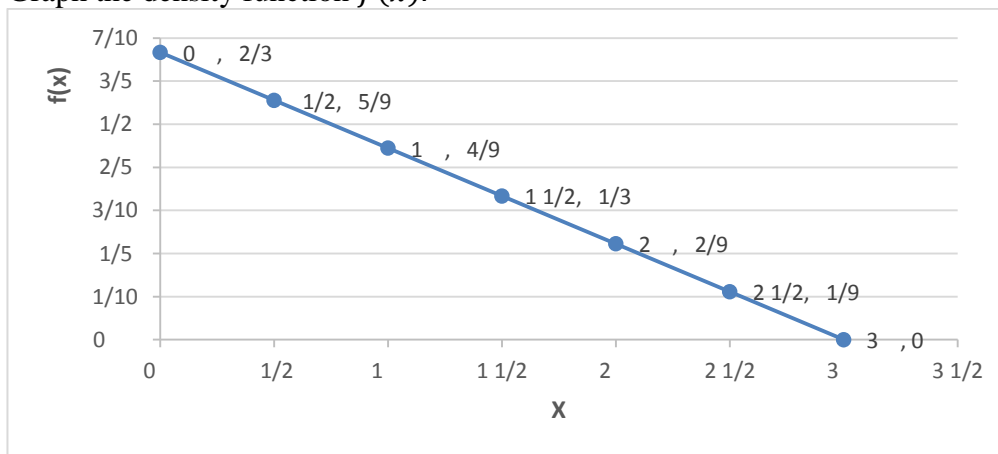
$$\sigma_P = \sqrt{205} = 14.32$$

The standard deviation of the portfolio is 14.32%.

9. Let X be a continuous random variable with probability density function given by

$$f(x) = \frac{2}{3} - \frac{2}{9}x, \quad 0 \leq x \leq 3$$

- a) Graph the density function $f(x)$.



- b) Find the total area beneath $f(x)$ for $0 \leq x \leq 3$.

$$\text{Area} = \frac{1}{2} * \text{base} * \text{height} = \frac{1}{2} * 3 * \frac{2}{3} = 1$$

The total area beneath $f(x)$ for $0 \leq x \leq 3$ is 1.

- c) What is the probability that $2 \leq x \leq 3$?

$$\text{Area} = \frac{1}{2} * \text{base} * \text{height} = \frac{1}{2} * 1 * \left(\frac{2}{9} - 0\right) = \frac{1}{2} * 1 * \frac{2}{9} = \frac{1}{9}$$

The probability that $2 \leq x \leq 3$ is 1/9 or 11.1%.

10. The final marks in a statistics course are normally distributed with a mean of 70 and a standard deviation of 10. The professor must convert all marks to letter grades. She decides that she wants 10% A's, 30% B's, 40% C's, 15% D's, and 5% F's. Determine the cutoffs (what the actual marks are) for each letter grade.

$$A: P(Z < z_{.10}) = 0.9; z_{.10} = 1.28; z_{.10} = \frac{x - \mu}{\sigma}; ; 1.28 = \frac{x - 70}{10} \quad x = 82.8\%$$

$$B: P(Z < z_{.40}) = 0.6; z_{.40} = .25; z_{.40} = \frac{x - \mu}{\sigma}; .25 = \frac{x - 70}{10}; x = 72.5\%$$

$$C: P(Z < -z_{.20}) = 0.2; -z_{.20} = -.84; -z_{.20} = \frac{x - \mu}{\sigma}; -.84 = \frac{x - 70}{10}; x = 61.6\%$$

$$D: P(Z < -z_{.05}) = 0.05; -z_{.05} = -1.645; -z_{.05} = \frac{x - \mu}{\sigma}; -1.645 = \frac{x - 70}{10}; x = 53.55\%$$

$$F \quad x < 53.55\%$$

MM 01/02/16