Circle: DGD 1 (Cameron) LAST NAME (in capitals):

DGD 2 (Chelsea) First name:

Marks: /13 Student number:

MAT 1348C (Prof. M. Šajna) — Fourth Homework Assignment
Due Thursday Feb. 25, 2016 by 9:00am (new due time!)

Instructions:

Print out this document and staple the pages. You may write on both sides of the paper or insert additional pages if necessary.

Submit a finished, presentable product. *Drafts and illegible papers will not be marked.* Show all relevant work to receive full credit.

Submit the assignment to your TA in the DGD or in the *submission box labeled MAT 1348C* in the Department of Mathematics and Statistics.

Circle the DGD you attend. Your marked paper will be returned to you in that DGD.

Late assignments will not be accepted.

Important note on academic integrity:

Students are permitted, and indeed encouraged, to discuss homework problems with others, but are not permitted to help each other write the final solutions (unless the assignment is explicitly announced as a group assignment). Once you understand a solution, you must write it out entirely by yourself. For each question, any help from other people must be clearly acknowledged, as well as any sources used (e.g. textbooks, websites, videos), if that source contains a solution to a very similar question, or a new method or idea that you used that was not in the course materials. Failure to follow these rules constitutes plagiarism (academic fraud). Note that if one student copies from the other, both students have committed academic fraud. If we believe plagiarism has occurred, the students will receive:

- a mark of 0 for the current assignment if this is the first offence;
- a mark of 0 for the whole assignment component of the course if this is the second offence.

Students are advised to carefully examine the *University Guidelines on Academic Integrity* — see

`http://web5.uottawa.ca/mcs-smc/academicintegrity/home.php`

as well as the *Course Policy on Plagiarism* — see

`http://mysite.science.uottawa.ca/msajna//teaching/plagiarism_policy.html`

Please sign below to confirm that you have read, understand, and will follow these guidelines. Only signed papers will have the mark recorded.

Student's signature:

Q2 Alternative proof

Case 1: A is a knight. Then B is telling the truth, so he must be a spy. But then C is also telling the truth, so he can't be a knave, a contradiction.

Case 2: A is a knave. Then B is lying, so he must be a spy, and C is telling the truth, and must be a knight.

Case 3: A is a spy. Then B is telling the truth, so he must be a knight. But then C is also telling the truth, so he can't be a knave, a contradiction.

We conclude that A must be a knave, B a spy, and C a knight □

1. Give a **proof by contradiction** of the following theorem:

Let n be an integer. Then at least one of the integers n and $n + 1348$ is not divisible by 3. [4pts]

Proof strategy:

We have to prove

P: "At least one of n and $n+1348$ is not divisible by 3."

We do this by assuming $\neg P$ and deriving a contradiction. Here,

$\neg P$: "Both n and $n+1348$ are divisible by 3."

Proof: Assume $n = 3k$ and $n + 1348 = 3\ell$ for some integers k and ℓ. Then

$1348 = (n+1348) - n = 3\ell - 3k = 3(\ell - k) = 3m$,

where $m = \ell - k$ is an integer. Hence 3 divides 1348, a contradiction (since $1348 = 3 \cdot 449 + 1$).

We conclude that $\neg P$ is false, and hence P is true: at least one of n and $n+1348$ is not divisible by 3. □

2. A spy (who can lie or tell the truth) arrives on the Island of Knights and Knaves. You encounter three people, A, B, and C, and you know that one is a knight, one a knave, and one a spy. Each of these three knows the type of the other two.

   A says: "I am the knight."

   B says: "A is not the knave."

   C says: "B is not the knave."

   Using a **proof by cases**, show that A must be the knave, B must be the spy, and C must be the knight. [4pts]

Proof.

Case 1: A is the knight. Then B is telling the truth, so he must be a spy. But then C is also telling the truth, so he can't be a knave, a contradiction.

Case 2: B is the knight. Then he is telling the truth, so A must be the spy. But then, again, C is telling the truth, so he can't be a knave, a contradiction.

Case 3: C is the knight. Then he is telling the truth, so B must be the spy, and A must be the knave, and both of them are lying.

Thus, the only possibility that works is that A is the knave, B is the spy, and C is the knight. □

True or false?

3. Let A, B, C be three subsets of the universal set. For each of the following, either **prove** the statement or give a **counterexample.** [5pts]

   (a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

   (b) If $A \cup B = A \cup C$, then ~~$A = B$~~ $B = C$

   Notes:

   - *$A \oplus B$ denotes the* **symmetric difference** *of sets A and B (also denoted $A \Delta B$),*

   $$A \oplus B = \{x \in \mathcal{U} : x \in A \ominus x \in B\}.$$

   - *A counterexample for (a) would consist of particular sets A, B, C such that $(A \oplus B) \oplus C \neq A \oplus (B \oplus C)$.*

(a) True - proof using a membership table

A	B	C	A⊕B	(A⊕B)⊕C	B⊕C	A⊕(B⊕C)
1	1	1	0	1	0	1
1	1	0	0	0	1	0
1	0	1	1	0	1	0
1	0	0	1	1	0	1
0	1	1	1	0	0	0
0	1	0	1	1	1	1
0	0	1	0	1	1	1
0	0	0	0	0	0	0

identical columns

(b) False - counterexample:

$A = \{1, 2\}$

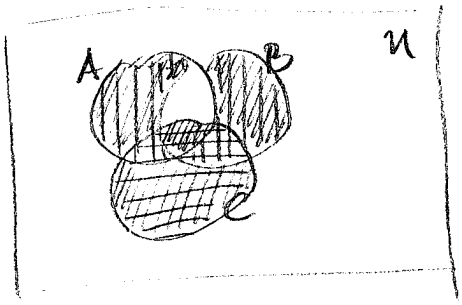
$B = \{3\}$

$C = \{2, 3\}$

Then $A \cup B = \{1, 2, 3\} = A \cup C$ but $B \neq C$.

(a) Alternative proofs

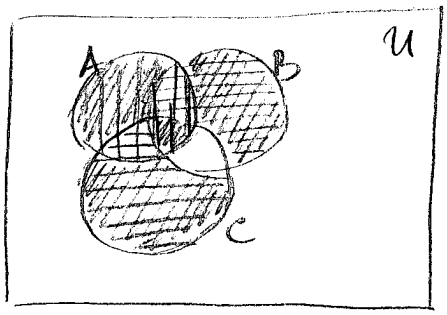
Using Venn diagrams



||| $A \oplus B$

$\equiv$ C

/// $(A \oplus B) \oplus C$



||| A

$\equiv$ $B \oplus C$

/// $A \oplus (B \oplus C)$

Using a rigorous proof (hard!)

(i) To prove $(A \oplus B) \oplus C \subseteq A \oplus (B \oplus C)$:

Take any $x \in (A \oplus B) \oplus C$. Then either $x \in A \oplus B$ or $x \in C$.

Case 1: $x \in A$. If $x \in C$, then $x \notin A \oplus B$, so $x \in B$ as well. Thus $x \notin B \oplus C$.

If $x \notin C$, then $x \in A \oplus B$, so $x \notin B$.

Hence $x \notin B \oplus C$.

Case 2: $x \notin A$. If $x \in C$, then $x \notin A \oplus B$, so $x \notin B$.

Hence $x \in B \oplus C$.

If $x \notin C$, then $x \in A \oplus B$, so $x \in B$.

Hence $x \in B \oplus C$.

We show that $x \in A \Leftrightarrow x \notin B \oplus C$. Hence $x \in A \oplus (B \oplus C)$.

It follows that $(A \oplus B) \subseteq A \oplus (B \oplus C)$.

(ii) To prove $A \oplus (B \oplus C) \subseteq (A \oplus B) \oplus C$:

similar to (i)...