University of Ottawa

Department of Mathematics and Statistics

MAT 1302F: Mathematical Methods II - Midterm 1 - Winter 16

Instructor: Xinhou Hua

Surname ____________________ First Name ____________________ Student # __________

Instructions:

(a) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(b) You are strongly recommended to write in **pen**, not pencil.
(c) You have to show your work for each question.
(d) All work to be considered for grading should be written in the space provided. If you have work on the back side, you should indicate this **clearly**. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) You have 80 minutes to complete this exam.

Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

____

Question	1	2	3	4	5	6	Total
Maximum	4	4	3	3	2	4	20
Grade							

1. [4 points] For each of the following statements, determine if it is true or false. For each correct answer, you will receive 1 point. For each incorrect answer, you will lose 1 point, but your total score for this question cannot be negative.

__________If the zero vector is a solution of the matrix equation $A\mathbf{x} = \mathbf{b}$, then this matrix equation is non-homogeneous.

Solution: False. If the zero vector is a solution then $b = Ax = A0 = 0$. So the equation is $Ax = 0$, thus homogenous.

__________The span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$.

Solution: True.

__________Let A be 5×5 matrix. If A has 5 pivot positions, then there exists vector $b \in \mathbb{R}^5$ such that the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Solution: False.

__________The vector $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ belongs to the Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Solution: True. $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

2. [4 points] Consider the following linear system

$$\begin{cases} x_1 \ + 2x_2 + 5x_4 &= 1 + x_3 \\ 2x_2 - 2x_3 + 6x_4 &= 3 - 2x_1 \\ x_1 \ + x_2 - x_3 &= 2 - 3x_4 \end{cases}$$

(a) **[3 points]** Row reduce the augmented matrix to the reduced row echelon form.

(b) **[1 point]** Determine if the linear system is consistent or inconsistent. (If the system is consistent, you do not need to find the general solution.)

Solution: (a)

We write the augmented matrix of the system and row reduce it

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 2 & 2 & -2 & 6 & 3 \\ 1 & 1 & -1 & 3 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & -1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0.5 \end{array}\right] \xrightarrow{R_3 \to 2R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 1 \\ 0 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ 0 & -2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 5 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(b) Since the rightmost column is a pivot column, the linear system is inconsistent.

3. [**3 points**] Find the general solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 \\ -2 & 4 & 3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

You should write your solution in **vector parametric form**.

Solution: We row reduce the augmented matrix of the corresponding linear system.

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 1 \\ -2 & 4 & 3 & 1 & -4 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 4 & -1 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right]$$

x_1, x_3 basic. x_2, x_4 free. Therefore the general solution is:

$$\begin{cases} x_1 = 2x_2 - 4x_4 - 1 \\ x_2 = \text{free} \\ x_3 = -3x_4 - 2 \\ x_4 = \text{free} \end{cases}$$

and the vector parametric form of the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 - 1 \\ x_2 \\ -3x_4 - 2 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

4. [**3 points**] Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 - x_3 &= 5 \\ x_1 + 2x_2 - x_3 &= 6 \\ 2x_1 + 2x_2 + kx_3 &= h. \end{cases}$$

For what values of h and k does the system have infinitely many solution?

Solution:

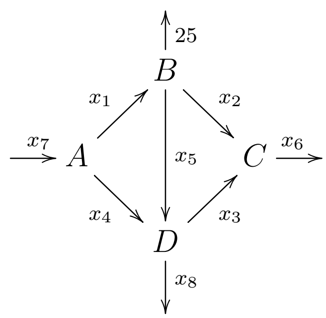
$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & 5 \\ 1 & 2 & -1 & 6 \\ 2 & 2 & k & h \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & k+2 & h-10 \end{bmatrix}.$$

To have infinitely many solutions, we should have

$$k + 2 = 0, \quad h - 10 = 0.$$

Hence $k = -2$, $h = 10$.

5. [2 points] Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter A or leave B, C and D during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



Write down a system of linear equations which describes the traffic flow. (**Do not solve the linear system.**)

Solution: Set "flow in" = "flow out" at each intersection:

$$\begin{aligned}
A: &\quad & x_7 &= x_1 + x_4 \\
B: &\quad & x_1 &= 25 + x_2 + x_5 \\
C: &\quad & x_2 + x_3 &= x_6 \\
D: &\quad & x_4 + x_5 &= x_8 + x_3 \\
Total: &\quad & x_7 &= 25 + x_6 + x_8
\end{aligned}$$

6. [**4 points**] Given four vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 8 \\ -2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 5 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -17 \\ 9 \end{bmatrix}$. Is $\mathbf{y}$ a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$? If yes, find the linear combination; if no, explain why.

Solution:

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 2 & 0 & 6 & 2 \\ -1 & 8 & 5 & -17 \\ 1 & -2 & 5 & 9 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 8 & -16 \\ 0 & -2 & 2 & 8 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & -2 & 2 & 8 \\ 0 & 8 & 8 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 4R_2} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & -2 & 2 & 8 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{(-1/2)R_2 \\ (1/16)R_3}} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 \to R_2 + R_3 \\ R_1 \to R_1 - 3R_3}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\mathbf{y}$ can be written as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$\mathbf{y} = -2\mathbf{u} - 3\mathbf{v} + \mathbf{w}.$$