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1. **(13 marks)** For the mechanical system shown in figure 1 where $F_a(t)$ is the known force applied to the top of the lever as shown (assuming the mass of the lever is negligible and that the motion of the lever is small):

   (a) Draw the free body diagram(s) that model the system.

   (b) Write the state-space representation, in matrix form, of the system by

      i. defining an appropriate set of state variables and

      ii. writing the set of first order state-space equations in matrix form.

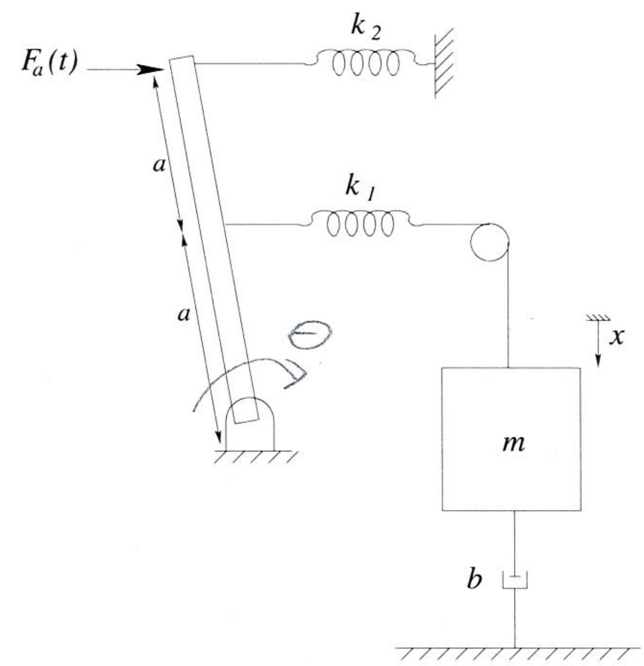

Figure 1: Mechanical system for question 1.

a)

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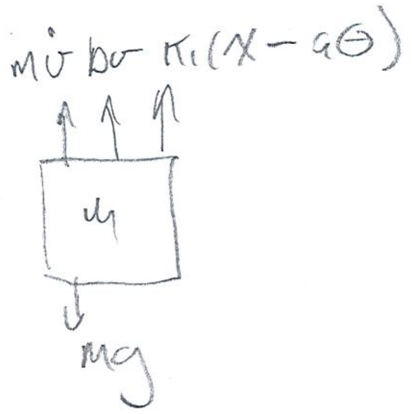

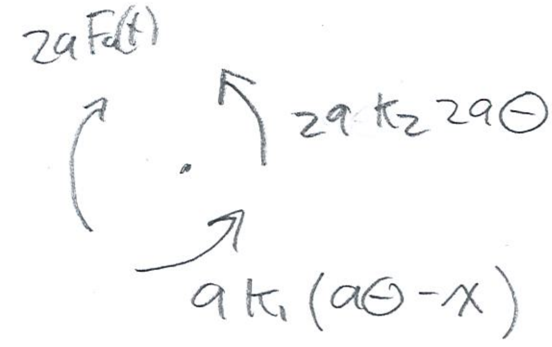

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b) i) define θ, x, v (of mass) as state variables

(ii) $\dot{x} = v$

$$\dot{v} = -\frac{b}{m}v - \frac{k_1}{m}x + \frac{k_1 a}{m}\theta + g$$

6

$$2a F_a(t) = 4a^2 k_2 \theta + a^2 k_1 \theta - a k_1 x$$

$$2a F_a(t) = a^2(4k_2 + k_1)\theta - a k_1 x$$

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$$\dot{\Theta} = \frac{K_1}{a(4K_2+K_1)} v + \frac{2}{a(4K_2+K_1)} \dot{F}_a(t)$$

so

$$\begin{bmatrix} \dot{\Theta} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{K_1}{a(4K_2+K_1)} \\ 0 & 0 & 1 \\ \frac{K_1 a}{m} & -\frac{K_1}{m} & -\frac{b}{M} \end{bmatrix} \begin{bmatrix} \Theta \\ x \\ v \end{bmatrix} + \begin{bmatrix} \frac{2\dot{F}_a(t)}{a(4K_2+K_1)} \\ 0 \\ g \end{bmatrix}$$

2. **(6 marks)** For a system modelled by the differential equation shown below where $x(t)$ is the response, show the numerical solution of the equation using the first order Euler approximation at $t = 0, t = 0.1$, and $t = 0.2$ for a time step of Δt=0.1 and given $x(0) = 0$.

$$10\dot{x}(t) + x(t) = 3$$

$\dot{x}(t) = \frac{3}{10} - \frac{x(t)}{10}$ 1

$t=0$ $x(0) = 0$ (given) 1

$t=0.1$ $x(0.1) = x(0) + \Delta t\, \dot{x}(0)$

$= 0 + 0.1\left(\frac{3}{10} - \frac{x(0)}{10}\right)$

2

$= 0.03$

$t=0.2$ $x(0.2) = x(0.1) + \Delta t\, \dot{x}(0.1)$

$= 0.03 + 0.1\left(\frac{3}{10} - \frac{x(0.1)}{10}\right)$

2

$= 0.03 + 0.1(0.3 - 0.003)$

$= 0.0597$

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3. **(12 marks)** For a system modelled by the following differential equation

$$4Dy(t) + 2y(t) = 5$$

   (a) What is the steady state solution?

   (b) What is the time constant, τ of the system?

   (c) Solve for and sketch the response, $y(t)$, given the initial value $y(0) = 10$.

   (d) Using the initial condition above, at what time, t_0, is the value of the response $y(t_0)$=5?

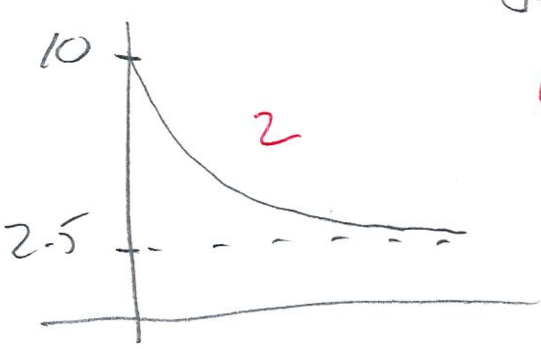
$$2Dy(t) + y(t) = \frac{5}{2}$$

a) $y_{ss} = \frac{5}{2}$ 2

b) $\tau = 2$ 2

$$y(t) = Ae^{-t/\tau} + B$$

c)



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4 $B = 2.5$

$A + B = 10 \rightarrow A = 7.5$

$$y(t) = 7.5e^{-t/2} + 2.5$$

d) $5 = 7.5e^{-t/2} + 2.5$

2 $\ln\left(\frac{2.5}{7.5}\right) = -t/2$

$t = 2.197$

4. **(17 marks)** The thermal characteristics of a travel mug for keeping coffee hot can be modelled as shown below in figure 2. The heating element that plugs into the car lighter socket is capable of delivering $Q(t)$=100 W of heat. The volume of the coffee is 300 ml, and the mug has a total surface area of 0.3 m^2.

   (a) Sketch the flow graph that models the system.

   (b) What is the time constant of the system?

   (c) Calculate the thermal capacitance, C, of the coffee (assume the properties of water).

   (d) You wish to calculate how much insulation is required for the mug.

      i. Given an ambient temperature of $T_a = 15$ °C and the thermal capacitance, C, calculated as above, calculate the thermal resistance, R, that will cause a temperature drop from an initial temperature of $T(0) = 85$ °C, to a temperature of 70 °C over a 10 minute period with no heating.

      ii. Given an insulating material with a thermal conductivity of $k_t = 0.03$ W/m·K, what is the required thickness of the insulation to achieve the above thermal resistance, R?

      iii. For the above results, is the heating element sufficient to maintain a coffee temperature of 80 °C given an ambient temperature of $T_a = 15$ °C? Justify your result.

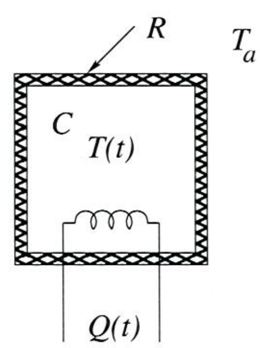
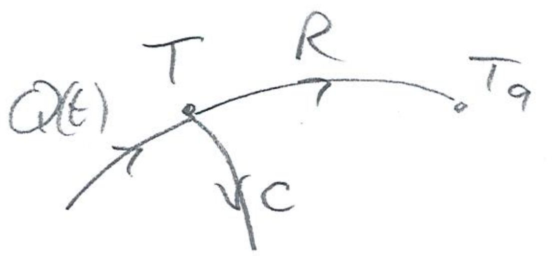


Figure 2: Thermal system for question 4.

a)



4

b) $Q(t) = CDT(t) + \frac{1}{R}(T(t) - T_a)$

$RQ(t) + T_a = RCDT(t) + T(t)$

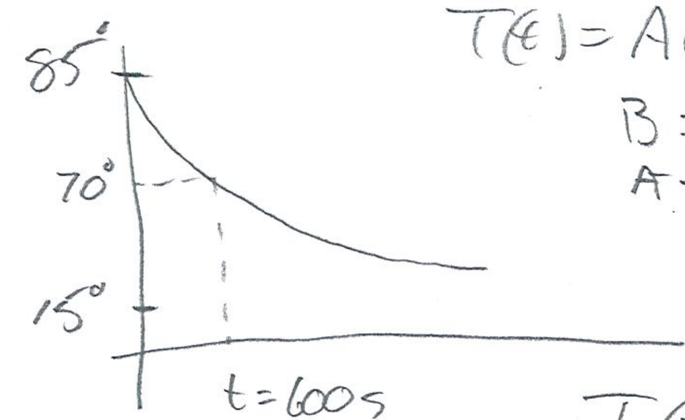
$\tau = RC$. 1

c) $C = mC_p$ $\quad m = 0.3$ Kg

$C_p = 4200 \frac{J}{Kg\,K}$

$C = 1260 \frac{J}{K}$ 2

d i)



$T(t) = Ae^{-t/\tau} + B$

$B = 15$

$A + B = 85$

$T(t) = 70e^{-t/\tau} + 15$

6

$70 = 70e^{-600/\tau} + 15$

$\ln\left(\frac{55}{70}\right) = -\frac{600}{\tau}$

$\tau = 2488\ s$

$2488\,s = R \cdot 1260\ \frac{J}{K}$

$R = 1.97\ \frac{K}{W}$

ii $R = \frac{L}{K_t A}$

2

$L = 1.97\ \frac{K}{W} \cdot 0.03\ \frac{W}{mK} \cdot 0.3\ m^2$

$= 0.0178\ m = 1.78\ cm$

iii $Q = \frac{1}{R}\Delta T$

2

$= \frac{1}{1.97} \cdot 65° = 33\ W$

$\therefore$ 100 W is sufficient

