Name & ID:

1. **(11 marks)** For the electrical system shown below in figure 1:

   (a) For each element, write the element law that relate the through and across variables.

   (b) Write the state-space representation of the system by

      i. defining an appropriate set of state variables and

      ii. writing the set of first order state-space equations.

   (c) Combine the above n first order state-space equations into a single nth order differential equation that relates the input voltage $e_i(t)$ and the voltage $e_o(t)$.

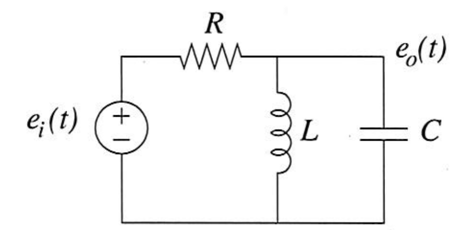

Figure 1: Electrical system for question 1.

a) $i_R = \frac{1}{R}(e_i - e_o)$

$i_C = CDe_o$

$i_L = \frac{1}{LD} e_o$

3

b) i) define e_o & i_L as state variables

2

ii) i_L:

$Di_L = \frac{1}{L} e_o$

e_o: $\frac{1}{R}(e_i - e_o) = i_L + CDe_o$

4

$De_o = -\frac{1}{C} i_L + \frac{1}{RC} e_i - \frac{1}{RC} e_o$

c) $\frac{1}{R}(e_i - e_o) = \frac{1}{LD} e_o + CDe_o$

$\frac{1}{R} De_i - \frac{1}{R} De_o = \frac{1}{L} e_o + CD^2 e_o$

2

Name & ID:

1. **(11 marks)** For the electrical system shown below in figure 1:

   (a) For each element, write the element law that relate the through and across variables.

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      i. defining an appropriate set of state variables and

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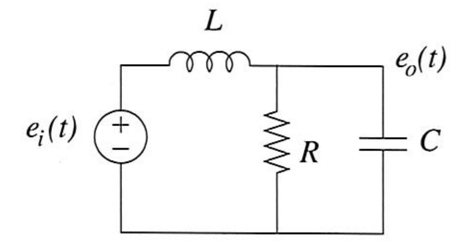


Figure 1: Electrical system for question 1.

a) $i_R = \frac{1}{R} e_o$

$i_C = C D e_o$ 3

$i_L = \frac{1}{LD}(e_i - e_o)$

b) i Define e_o & i_L as state variables 2

ii i_L:

$D i_L = \frac{1}{L} e_i - \frac{1}{L} e_o$ 4

e_o: $i_L = i_R + i_C = \frac{1}{R} e_o + C D e_o$

$D e_o = \frac{1}{C} i_L - \frac{1}{RC} e_o$

c) $i_L = \frac{1}{DL}(e_i - e_o)$

$\frac{1}{L} e_i - \frac{1}{L} e_o = \frac{1}{R} D e_o + C D^2 e_o$ 2

Name & ID:

2. **(6 marks)** For a system modelled by the differential equation shown below where $x(t)$ is the response, show the numerical solution of the equation using the first order Euler approximation at $t = 0, t = 0.1$, and $t = 0.2$ for a time step of Δt=0.1 and given $x(0) = 10$.

$$10\dot{x}(t) + x(t) = 0$$

$\dot{x}(t) = \frac{-1}{10} x(t)$

$x(0) = 10$

$x(0.1) = x(0) + \Delta t \, \dot{x}(0)$

$= 10 + 0.1\left(-\frac{1}{10} \times 10\right)$

$= 9.9$

$x(0.2) = x(0.1) + \Delta t \, \dot{x}(0.1)$

$= 9.9 + 0.1\left(\frac{-1}{10} \times 9.9\right)$

$= 9.801$

6

2. **(6 marks)** For a system modelled by the differential equation shown below where $x(t)$ is the response, show the numerical solution of the equation using the first order Euler approximation at $t = 0, t = 0.1$, and $t = 0.2$ for a time step of Δt=0.1 and given $x(0) = 10$.

$$8\dot{x}(t) + x(t) = 0$$

$$\dot{x} = -\frac{1}{8} x$$

$$x(0) = 10$$

$$x(0.1) = x(0) + \Delta t \, \dot{x}(0)$$

$$= 10 + 0.1 \left(-\frac{1}{8} \cdot 10\right)$$

$$= 9.875$$

$$x(0.2) = x(0.1) + \Delta t \, \dot{x}(0.1)$$

$$= 9.875 + 0.1 \left(-\frac{1}{8} \cdot 9.875\right)$$

$$= 9.752$$

6

Name & ID:

3. **(12 marks)** For a system modelled by the following differential equation

$$4Dy(t) + 2y(t) = 5$$

   (a) What is the steady state solution?

   (b) What is the time constant, τ of the system?

   (c) Solve for and sketch the response, $y(t)$, given the initial value $y(0) = 10$.

   (d) Using the initial condition above, at what time, t_0, is the value of the response $y(t_0)$=5?

$$2Dy(t) + y(t) = \frac{5}{2}$$

a) $y_{ss} = \frac{5}{2}$ 2

2

b) $\tau = 2$

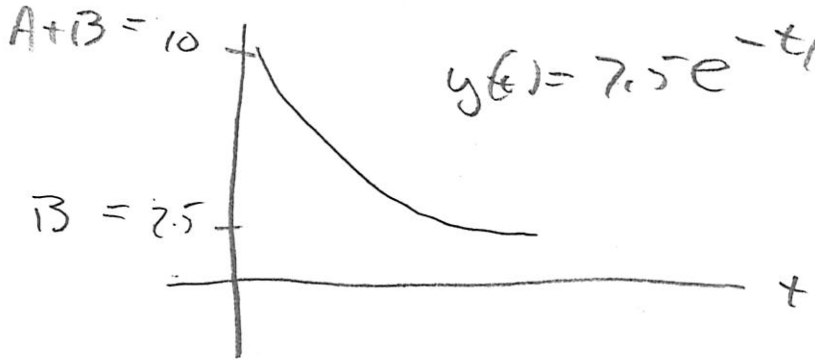
c) $y(t) = Ae^{-t/2} + B$

$y(0) = A + B = 10$

$y_{ss} = B = \frac{5}{2}$

$A = 7.5$

$y(t) = 7.5e^{-t/2} + 2.5$ 6



d) $5 = 7.5e^{-t/2} + 2.5$ 2

$-\frac{t}{2} = \ln\left(\frac{2.5}{7.5}\right)$

$t = 2.2$

Name & ID:

3. **(12 marks)** For a system modelled by the following differential equation

$$4Dy(t) + 3y(t) = 6$$

   (a) What is the steady state solution?

   (b) What is the time constant, τ of the system?

   (c) Solve for and sketch the response, $y(t)$, given the initial value $y(0) = 10$.

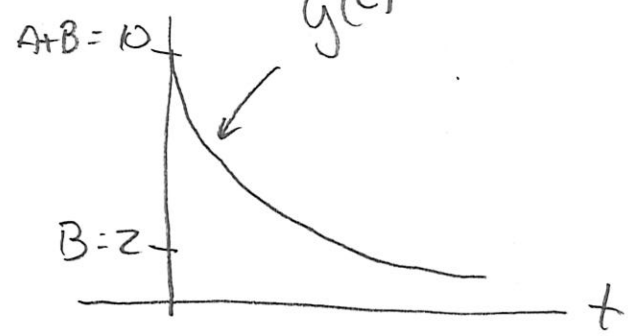
   (d) Using the initial condition above, at what time, t_0, is the value of the response $y(t_0)$=5?

$$\frac{4}{3} D\, y(t) + y(t) = 2.$$

a) $y_{ss} = 2$ (2)

b) $\tau = \frac{4}{3}$ (2)

c) $y(t) = 8e^{-t\,3/4} + 2$



$y(t) = Ae^{-t/\tau} + B$

$y(0) = A + B$

$y_{ss} = B$

$A = 10$

$B = 2$ (6)

d) $5 = 8e^{-t\frac{3}{4}} + 2$ (2)

$-\frac{t\,3}{4} = \ln\frac{3}{8}$

$t = 1.3$

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4. **(16 marks)** Figure 2 below shows an irrigation system with a constant input flow of water of $Q_i(t) = 50 \times 10^{-6}$ m^3/s into a cylindrical tank having a diameter of 2.2 m and a height of 5 m.

   (a) When the tank is filled to a height of 4 m the valve is opened and the initial outflow at the end of the pipe is $Q_o(0) = 200 \times 10^{-6}$ m^3/s. What is a reasonable value to use as the resistance R that models the total hydraulic resistance of the outflow pipe and valve?

   (b) What is the hydraulic capacitance, C, of the tank?

   (c) What is the differential equation that relates the pressure at the bottom of the tank with appropriate known values? (Note: use numerical values where possible)

   (d) Given the above initial height of water in the tank, find the outflow 30 minutes after the valve opens, i.e., find $Q_o(t = 30\,\text{min})$.

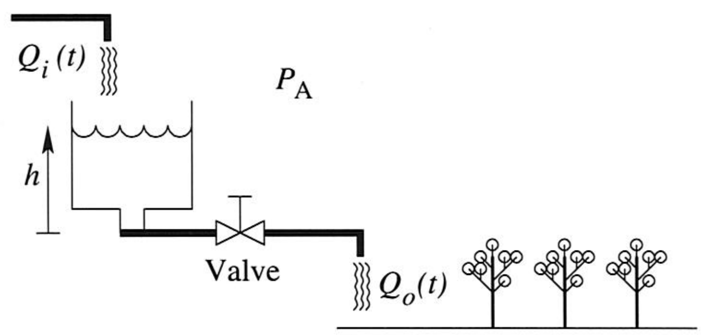


Figure 2: Hydraulic system for question 4.

a) $P = h\rho g \ (+P_A)$

$= 4m \times 9.8 \frac{m}{s^2} \times 1000 \frac{Kg}{m^3}$

$= 39.2 \times 10^3 \frac{Kg}{m s^2} = \frac{Kg\, m}{s^2}/m^2 = \frac{N}{M^2} = Pa \ (+P_A)$

$Q_o = \frac{1}{R}(P\ (-P_A))$

$R = \frac{P}{Q_o} = \frac{39.2 \times 10^3\ Pa}{200 \times 10^{-6} \frac{m^2}{s}}$

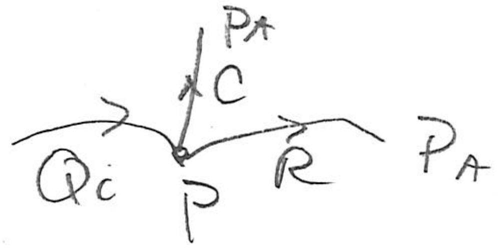
$= 196 \times 10^6 \frac{Pa\, s}{m^3}$

4

b) $C = \frac{A}{\rho g} = \frac{\pi (1.1)^2 m^2}{1000 \frac{Kg}{m^3} \times 9.8 \frac{m}{s^2}}$

2

$= 387.9 \times 10^{-6} \frac{m^4 s^2}{Kg} = \frac{m^3}{Pa}$

c) 

$Q_i = CDP + \frac{1}{R}P$ (P relative to P_A)

4 $RQ_i = RCDP + P$

$\tau = RC = 196\times10^6 \frac{Pa\,s}{m^3} \times 387.9\times10^{-6}\frac{m^3}{Pa}$

$= 76\times10^3\,s$

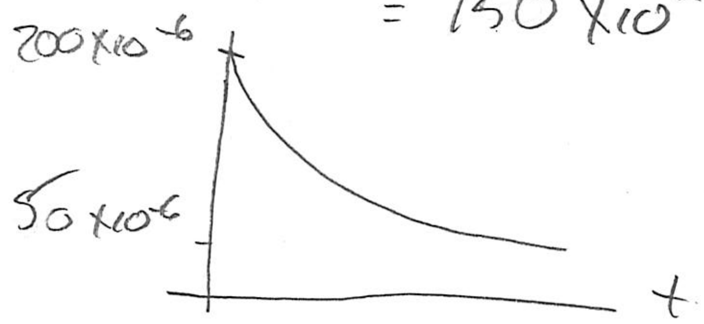
$196\times10^6\frac{Pa\,s}{m^3} \times 50\times10^{-6}\frac{m^3}{s} = 76\times10^3 DP + P$

d) $Q_o(0) = 200\times10^{-6}\frac{m^3}{s} = A + B$

$Q_{o\,ss} = 50\times10^{-6}\frac{m^3}{s} = B$

$Q_o(t) = Ae^{-t/\tau} + B$

$= 150\times10^{-6}e^{-t/\tau} + 50\times10^{-6}$



at $t = 30$ min $= 1800$ s

6 $Q_o = 150\times10^{-6}e^{-1800/5700} + 50\times10^{-6}$

$= 196.5\times10^{-6}\frac{m^3}{s}$

Name & ID:

4. **(16 marks)** Figure 2 below shows an irrigation system with a constant input flow of water of $Q_i(t) = 60 \times 10^{-6}$ m³/s into a cylindrical tank having a diameter of 1.8 m and a height of 5 m.

   (a) When the tank is filled to a height of 4.5 m the valve is opened and the initial outflow at the end of the pipe is $Q_o(0) = 200 \times 10^{-6}$ m³/s. What is a reasonable value to use as the resistance R that models the total hydraulic resistance of the outflow pipe and valve?

   (b) What is the hydraulic capacitance, C, of the tank?

   (c) What is the differential equation that relates the pressure at the bottom of the tank with appropriate known values? (Note: use numerical values where possible)

   (d) Given the above initial height of water in the tank, find the outflow 30 minutes after the valve opens, i.e., find $Q_o(t = 30\,\mathrm{min})$.

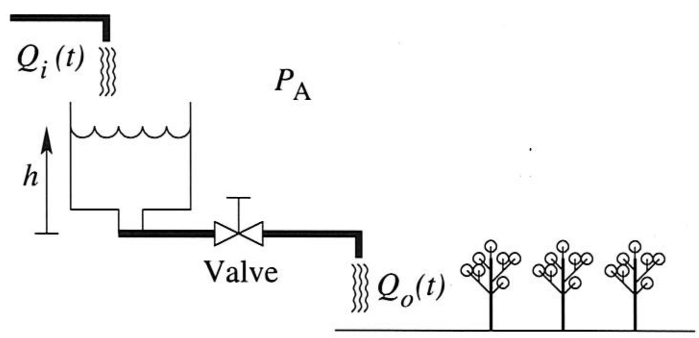


Figure 2: Hydraulic system for question 4.

a) $P = h\rho g \;(+P_A)$

$= 4.5\,m \times 9.8\,\frac{m}{s^2} \times 1000\,\frac{kg}{m^3}$

$= 44.1 \times 10^3\,\frac{kg}{m s^2} = kg\frac{m}{s^2}/m^2 = \frac{N}{m^2} = Pa \quad (+P_A)$

$Q_o = \frac{1}{R}(P(-P_A))$

$R = \frac{P}{Q_o} = \frac{44.1 \times 10^3\,Pa}{200 \times 10^{-6}\,\frac{m^3}{s}}$ 4

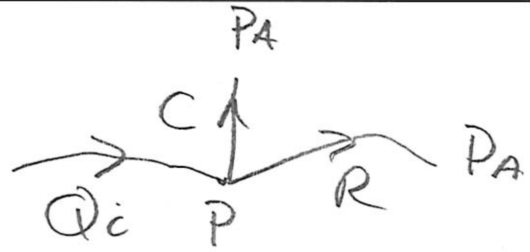
$= 220.5 \times 10^6\,\frac{Pa\,s}{m^3}$

b) $C = \frac{A}{\rho g} = \frac{\pi (0.9)^2\,m^2}{1000\,\frac{kg}{m^3} \times 9.8\,\frac{m}{s^2}}$ 2

$= 259.7 \times 10^{-6}\,\frac{m^4 s^2}{kg} = \frac{m^3}{Pa}$

Name & ID:

c)



$Q_i = CDP + \frac{1}{R}P$ (P relative to P_A)

(4) $RQ_i = RCDP + P$

$\tau = RC = 220.5\times10^{6}\ \frac{Pa\,s}{m^3} \times 259.7\times10^{-6}\ \frac{m^3}{Pa}$

$= 57\times10^{3}\ s$

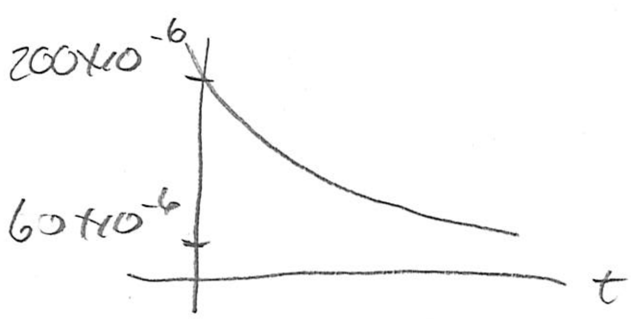
$220.5\times10^{6}\ \frac{Pa\,s}{m^3} \times 60\times10^{-6}\ \frac{m^3}{s} = 57\times10^{3}\ s\ DP + P$

d) $Q_o(0) = 200\times10^{-6}\ \frac{m^3}{s} = A+B$

$Q_{o\,ss} = 60\times10^{-6}\ \frac{m^3}{s} = B$

$Q_o(t) = Ae^{-t/\tau} + B$

$\therefore Q_o(t) = 140\times10^{-6}e^{-t/\tau} + 60\times10^{-6}$



(6)

at $t = 30\ min = 1800\ s$

$Q_o = 140\times10^{-6}e^{-1800/57\times10^{3}} + 60\times10^{-6}$

$= 195.4\times10^{-6}\ \frac{m^3}{s}$

