Short-answer questions — write your final answer in the answer box. Wherever indicated, you must briefly justify your answers to receive full marks.

[2pts] 1. The truth table of a compound proposition p with atomic propositions A, B, and C is as follows:

A	B	C	p
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Give a **disjunctive normal form** of p. *Do not simplify your answer.*

DNF: $(A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$

No justification is needed.

[2pts] 2. Let P_1, P_2, P_3 be propositions. Complete the following definition:

A set $\{P_1, P_2, P_3\}$ is said to be **consistent** if

$P_1 \wedge P_2 \wedge P_3$ is not a contradiction

No justification is needed.

[3pts] 3. Give an example of a proposition in propositional variables a and b that is:

- a tautology: $(a \wedge b) \vee \neg(a \wedge b)$
- a contradiction: $(a \wedge b) \wedge \neg(a \wedge b)$
- a contingency $a \wedge b$

No justification is needed.

[2pts] 4. Consider the following atomic propositions:

N: "The system is operating normally."

M: "Messages are scanned for viruses."

U: "Software is being updated."

F: "Users can access the file system. "

Translate the following sentence into a compound proposition using propositional variables N, M, U, and F. *Parenthesis are included to clarify the structure.*

[[The system operating normally and software not being updated] is necessary and sufficient for users to be able to access the file system] only if messages are not scanned for viruses.

Compound proposition: $((N \wedge \neg U) \leftrightarrow F) \rightarrow \neg M$

No justification is needed.

5. On the Island of Knights and Knaves, as you know, there are two types of natives, indistinguishable by sight: knights, who always tell the truth, and knaves, who always lie. [4pts]

   Strolling on the island, we meet two inhabitants A and B. Person A says: "Both of us are knaves." What can you conclude about the types (knight or knave) of persons A and B?

Answer: A is a knave, B is a knight.

Justification:

Define a: "A is a knight."
b: "B is a knight."

A says: $\neg a \wedge \neg b$

a	b	$\neg a \wedge \neg b$	
T	T	F	X
T	F	F	X
F	T	F	✓
F	F	T	X

We need the same truth value for a and $\neg a \wedge \neg b$.

Hence $a = F$ and $b = T$

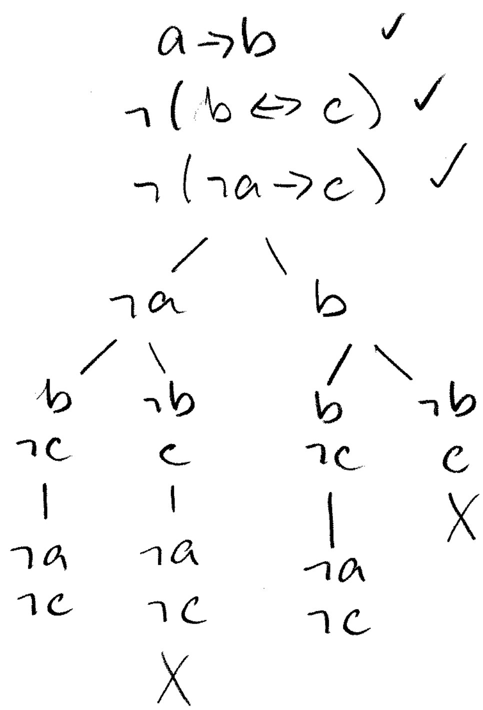
[4pts] 6. Is the following argument valid? If not, give a counterexample.

$$\begin{array}{l} a \to b \\ \neg(b \leftrightarrow c) \\ \hline \therefore \neg a \to c \end{array}$$

Answer: The argument is (circle): valid **invalid**

Counterexample (if applicable): a=F, b=T, c=F

Justification:



Or:

a	b	c	$a \to b$	$\neg(b \leftrightarrow c)$	$\neg a \to c$	
T	T	T	T	F	T	
T	T	F	T	T	T	
T	F	T	F	T	T	
T	F	F	F	F	T	
F	T	T	T	F	T	
F	T	F	T	T	F	← counterexample
F	F	T	T	T	T	
F	F	F	T	F	F	

7. Consider an argument with premises P_1 and P_2, and conclusion C:

$$\begin{array}{l} P_1 \\ \underline{P_2} \\ \therefore C \end{array}$$

[4pts] Which of the following statements about the above argument are true (for all propositions P_1, P_2, and C)?

(a) If $P_1 \wedge P_2 \wedge \neg C$ is a contradiction, then the argument is valid.

(b) If the argument is valid, then $P_1 \vee P_2 \to C$ is a tautology.

(c) If $\{P_1, P_2\}$ is inconsistent, then the argument is valid.

(d) If the argument is valid, then C is true.

Answer: The **true** statement(s) are *(list the letters)*: a, c

The **false** statement(s) are *(list the letters)*: b, d

No justification is needed.

8. Let P be a complex proposition, and consider a complete truth tree with P in the root.
[3pts] Which of the following statements are true (for all propositions P)?

(a) If the truth tree for P has no closed paths, then P is a tautology.

(b) If the truth tree for P has no open paths, then $\neg P$ is a tautology.

(c) The number of complete open paths is equal to the number of counterexamples to the statement "P is a contradiction".

Answer: The **true** statement(s) are *(list the letters)*: b

The **false** statement(s) are *(list the letters)*: a, c

No justification is needed.

Q7 – explanation

(a) $P_1 \land P_2 \land \neg C \equiv \neg(P_1 \land P_2 \to C)$

The argument is valid if and only if $P_1 \land P_2 \to C$ is a tautology, and hence if and only if $P_1 \land P_2 \land \neg C$ is a contradiction.

(b) If the argument is valid, $P_1 \land P_2 \to C$ is a tautology, but $P_1 \lor P_2 \to C$ need not be (i.e. it may be that C is false if only one of P_1 and P_2 is true)

(c) If $\{P_1, P_2\}$ is inconsistent, then $P_1 \land P_2$ is a contradiction, and $P_1 \land P_2 \to C$ is always true.

(d) C need not be true if $P_1 \land P_2$ is false.

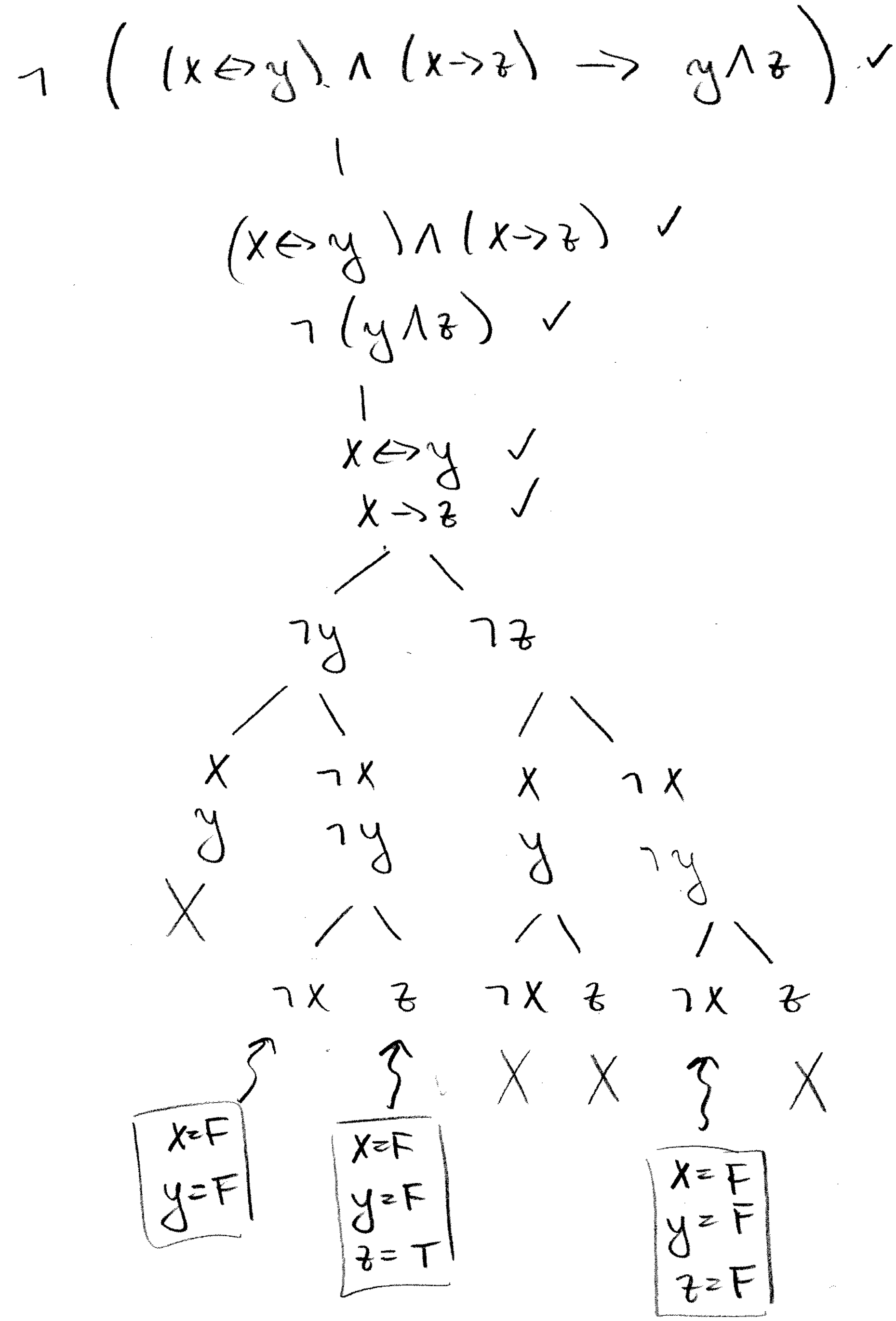
Q8 – explanation

(a) If the truth tree for P has no closed paths, then P is not a contradiction, but it need not be a tautology.

(b) In this case, P is a contradiction, so $\neg P$ is a tautology.

(c) Each open path represents one or more counter-examples, and distinct paths may represent the same counterexample(s).

[6pts] 10. Let P be the proposition $((x \leftrightarrow y) \wedge (x \rightarrow z)) \rightarrow (y \wedge z)$. Using an **appropriate truth tree**, determine whether or not P is a tautology. If the answer is negative, give **all counterexamples**.



Answer: The proposition is a tautology (circle): True **False**

All counterexamples (if applicable): $x = F,\ y = F,\ z = T$

$x = F,\ y = F,\ z = F$

Long-answer questions. Detailed solutions are required.

5 [~~4~~pts] 9. Prove the following using only the equivalences listed on Page 11. Justify each step by giving the number of the corresponding equivalence on Page 11. *Do not skip steps or combine several equivalences into a single step. Do not omit parentheses.*

$$(a \vee b) \wedge \Big((a \to c) \wedge (b \to c)\Big) \equiv (a \vee b) \wedge c$$

$(a \vee b) \wedge \big((a \to c) \wedge (b \to c)\big)$ (1)

$\equiv (a \vee b) \wedge \big((\neg a \vee c) \wedge (\neg b \vee c)\big)$ (13)

$\equiv (a \vee b) \wedge \big((c \vee \neg a) \wedge (c \vee \neg b)\big)$ (17)

$\equiv (a \vee b) \wedge \big(c \vee (\neg a \wedge \neg b)\big)$ (20)

$\equiv (a \vee b) \wedge \big(c \vee \neg(a \vee b)\big)$ (18)

$\equiv \big((a \vee b) \wedge c\big) \vee \big((a \vee b) \wedge \neg(a \vee b)\big)$ (5)

$\equiv \big((a \vee b) \wedge c\big) \vee \mathbf{F}$ (6)

$\equiv (a \vee b) \wedge c$

11. *Recall that an integer m divides an integer n if $n = km$ for some integer k.*

[5pts]

Let m and n be positive integers. Give an **indirect proof** of the following theorem. *(Parentheses are included to clarify the structure of the statement.)*

Theorem: If n is odd, then [m is odd or m does not divide n].

Strategy: Define p: "n is odd"
q: "m is odd"
r: "m divides n"

We have to prove $p \to (q \vee \neg r)$ using an indirect proof. Hence, we assume $\neg(q \vee \neg r)$ and show that $\neg p$ follows. Note that $\neg(q \vee \neg r) \equiv \neg q \wedge r$.

Proof. Assume m is even and m divides n.
Hence $m = 2k$ for some integer k, and $n = lm$ for some integer l. Then:

$$n = lm = l(2k) = 2(lk) = 2t \quad \text{for } t = lk.$$

Since l and k are integers, so is t.
Hence n is even.

Conclusion: We proved $(\neg q \wedge r) \to \neg p$ directly. This gives an indirect proof of $p \to (q \vee \neg r)$