

Solution:

HW #2

1. (3 points) Prove the following sequence is convergent, and find its limit, using theorems from class. A rigorous and well-written answer is required.

$$a_n = \frac{n4^{3n} + 5}{6n5^{4n}} \quad (n \geq 1).$$

Solution:

We write

$$a_n = \frac{n(4^3)^n}{6n(5^4)^n} + \frac{5}{6n(5^{4n})} = \frac{1}{6} \left(\frac{64}{625} \right)^n + \frac{5}{6n(5^{4n})} = b_n + c_n$$

Let $\varepsilon > 0$. Since $0 < \frac{64}{625} < 1$, we know that there exists $N \in \mathbb{N}$ such that $\left(\frac{64}{625}\right)^N < 6\varepsilon$. Therefore, for every $n \geq N$, we have

$$0 < \frac{1}{6} \left(\frac{64}{625} \right)^n \leq \frac{1}{6} \left(\frac{64}{625} \right)^N < \varepsilon,$$

whence for all $n \geq N$, $|b_n - 0| = b_n < \varepsilon$, and so $\{b_n\}_{n \in \mathbb{N}}$ converges to 0.

(Alternately: acceptable to cite as a fact that if $|r| < 1$ then $r^n \rightarrow 0$, and use theorem on arithmetic of convergent sequences to insert the scalar multiple.)

Since $5^{4n} \geq 1$ for all n , we have

$$0 < c_n < \frac{5}{6n}$$

for all n . By the theorem on the arithmetic of convergent sequences, we know that since $\frac{1}{n} \rightarrow 0$, then $\frac{5}{6} \left(\frac{1}{n}\right) \rightarrow \frac{5}{6}(0) = 0$. Therefore by the squeeze theorem, $c_n \rightarrow 0$.

By the theorem on the arithmetic of convergent sequences, since b_n and c_n converge (to 0), it follows that their sum a_n also converges to 0.

2. (5 points) Let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence defined by $a_0 = 1$ and $a_{n+1} = \sqrt{3a_n + 4}$ for all $n \geq 0$.
 (a) Prove by induction that $0 \leq a_n \leq 4$ for all $n \in \mathbb{N}$.

Solution:

Base case: it is true that $0 \leq a_1 = 1 \leq 4$.

Induction hypothesis: let us suppose that for a certain $n \geq 1$, we have $0 \leq a_n \leq 4$.

We need to prove that $0 \leq a_{n+1} \leq 4$.

Since $0 \leq a_n \leq 4$, we have $0 \leq 3a_n + 4 \leq 3(4) + 4 = 16$. Taking square roots yields $0 \leq \sqrt{3a_n + 4} \leq 4$, whence $0 \leq a_{n+1} \leq 4$, as required.

Therefore, by the principle of mathematical induction, this statement holds for all integers $n \geq 1$.

- (b) Prove that the sequence $\{a_n\}_{n \in \mathbb{N}}$ is increasing.

Solution:

We compute

$$\begin{aligned} a_{n+1} - a_n &= \sqrt{3a_n + 4} - a_n \\ &= \frac{(3a_n + 4) - a_n^2}{\sqrt{3a_n + 4} + a_n} \quad \text{by rationalizing} \\ &= \frac{-(a_n + 1)(a_n - 4)}{\sqrt{3a_n + 4} + a_n} \end{aligned}$$

When $0 \leq a_n \leq 4$, $a_n - 4 \leq 0$ and $a_n + 1 > 0$ so the numerator is nonnegative; moreover, the denominator is always positive if $a_n \geq 0$. Therefore $a_{n+1} - a_n \geq 0$, whence $a_{n+1} \geq a_n$ for all n . Thus the sequence is increasing.

- (c) Explain why $\{a_n\}_{n \in \mathbb{N}}$ is convergent, and find its limit.

Solution:

The sequence is increasing, by (b), and bounded above by 4 (as shown in (a)). Therefore the sequence is convergent.

Let a be its limit. Then $a = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1}$. But $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{3a_n + 4}$. Since $\sqrt{3x + 4}$ is a continuous function, we have

$$a = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{3a_n + 4} = \sqrt{3a + 4}$$

whence $a^2 = 3a + 4$ so that $a \in \{-1, 4\}$. Since $a_n \in [0, 4]$ for all n , we know that $a \in [0, 4]$ as well, by a theorem from class, therefore it must be the case that $a = 4$.

3. (6 points) Prove that the following functions are continuous on their domains, using the methods and results from MAT1325:

(a) $f(x) = |5x + 1|$: prove this directly from the definition.

Solution:

Let $c \in \mathbb{R}$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers converging to c . We need to prove that $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to $f(c)$.

Let $\varepsilon > 0$. Then since $x_n \rightarrow c$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \varepsilon/5$. So let $n \geq N$; then

$$\begin{aligned} |f(x_n) - f(c)| &= ||5x_n + 1| - |5c + 1|| \\ &\leq |(5x_n + 1) - (5c + 1)| \quad \text{by a version of the triangle inequality shown in the DGD; see below} \\ &= 5|x_n - c| < \varepsilon \end{aligned}$$

as required.

Solution:

Triangle inequality: The triangle inequality states that for all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$. Let $x = a - b$ and $y = b$. Then $x + y = a$, so we have

$$|a| \leq |a - b| + |b|$$

which gives $|a| - |b| \leq |a - b|$. Interchanging a and b yields $|b| - |a| \leq |b - a| = |a - b|$. Since having $z \leq t$ and $-z \leq t$ implies $|z| \leq t$, we conclude that for all $a, b \in \mathbb{R}$

$$||a| - |b|| \leq |a - b|.$$

(b) Applying only results stated or proven in class :

$$g(x) = \begin{cases} 3 \cos(x\pi) & \text{if } x > -2; \\ 3x^2 - 9 & \text{if } x \leq -2. \end{cases}$$

(Your logical argument is required: you must explain how to use the theorems we saw in class to prove the continuity of this function.)

Solution:

In class, we stated a theorem that the functions x and $\cos(x)$ are continuous. We also proved that the sum, product, and composition of continuous functions are continuous. Therefore the functions $g(x) = 3 \cos(x\pi)$ and $h(x) = 3x^2 - 9$ are continuous on their domains. Furthermore, we proved that restricting a function to a smaller domain which is an open interval preserves its continuity; therefore on $(2, \infty)$ we have $f(x) = g(x) = 3 \cos(x\pi)$ is continuous, and on $(-\infty, 2)$, $f(x) = h(x) = 3x^2 - 9$ is continuous.

It remains only to consider the continuity at $x = -2$. We use the theorem stated in class, that the limit exists iff the left and right limits coincide.

Since $g(x)$ is continuous at -2 , we know that $\lim_{x \rightarrow -2} g(x) = g(-2) = 3$. Therefore $\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} g(x) = 3$.

Since $h(x)$ is continuous at -2 , we know that $\lim_{x \rightarrow -2} h(x) = h(-2) = 3$. Therefore $\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} h(x) = 3$.

4. (4 points)

- (a) Prove that if f, g, h are three functions defined in a neighbourhood $(a-r, a+r)$ of a such that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = g(a)$ and for all $x \in (a-r, a+r)$, we have $f(x) \leq g(x) \leq h(x)$, then g is continuous at a .

Solution:

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers in the interval $(a-r, a+r)$ which converges to a . Then since $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = g(a)$, the definition of the limit of a function tells us that

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} h(x_n) = g(a).$$

Now for each $n \in \mathbb{N}$, we have by hypothesis that $f(x_n) \leq g(x_n) \leq h(x_n)$. Therefore by the squeeze theorem, we may conclude that

$$\lim_{n \rightarrow \infty} g(x_n) = g(a),$$

whence g is continuous at a .

- (b) Suppose f is a function such that on the interval $(-1, 1)$ we have $|f(x)| \leq x$, deduce using (a) that $f(x)$ is continuous at 0. (Note major typo: it should say $|f(x)| \leq |x|$.)

Solution:

The hypothesis implies that $-|x| \leq f(x) \leq |x|$ on the interval $(0-1, 0+1)$. In particular, we have $0 \leq f(0) \leq 0$, so that $f(0) = 0$. Since $\lim_{x \rightarrow 0} |x| = \lim_{x \rightarrow 0} -|x| = 0 = f(0)$, the hypotheses of (a) are all satisfied, whence we may conclude that f is continuous at 0.

5. (4 points)

- (a) Given an example of a continuous function f whose domain contains an interval $[a, b]$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[a, b]$ such that the sequence $\{f(x_n)\}_{n \in \mathbb{N}}$ converges but $\{x_n\}_{n \in \mathbb{N}}$ diverges.

Solution:

We just need to pick a bounded divergent sequence, like $(-1)^n$, which lies in the interval $[-1, 1]$, and a constant function, like $f(x) = 7$. Then $t = 7$ is such that $f(x_n) \rightarrow t$.

The point: the hypothesis certainly does not give you any control over the behaviour of $\{x_n\}_{n \in \mathbb{N}}$.

Solution:

0.5 points for each of : f continuous, x_n divergent, $x_n \in [a, b]$, $f(x_n)$ convergent

- (b) Suppose f is a continuous function whose domain includes an interval $[a, b]$. Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers in $[a, b]$ such that $f(x_n) \rightarrow t$. Prove that there exists a subsequence of $\{x_n\}_{n \in \mathbb{N}}$ that converges to a limit $c \in [a, b]$ for which $f(c) = t$.

Solution:

Since $\{x_n\}_{n \in \mathbb{N}}$ is a bounded sequence, by the Bolzano-Weierstrass theorem, it has a convergent subsequence (perhaps many). Choose one, call it $\{x_{n_k}\}_{k \in \mathbb{N}}$ and let c be its limit. By our theorem from class, we know the limit also lies in $[a, b]$. Since the sequence $\{f(x_n)\}_{n \in \mathbb{N}}$ converges to t , so does any subsequence; in particular, $f_{x_{n_k}} \rightarrow t$ as well. Since f is continuous, we must have $f(c) = t$.

Solution:

1 point Bolzano-Weierstrass, 1 point for applying f to the subsequence, 1 point for putting it all together

6. (Bonus) Continuity is usually defined by the following “ $\varepsilon - \delta$ criterion”:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x \in U \text{ s.t. } |x - c| < \delta, |f(x) - f(c)| < \varepsilon.$$

(a) Suppose that $f: U \rightarrow \mathbb{R}$ is a function that satisfies the $\varepsilon - \delta$ criterion at c (i.e. as written above). Prove that f is continuous at c (using our definition of continuity from class).

Solution:

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers in U which converges to c . Let $\varepsilon > 0$. By the criterion above, there is some value $\delta > 0$ such that if $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$.

Now since $x_n \rightarrow c$, and $\delta > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \delta$.

It follows from the criterion therefore that for all $n \geq N$, $|f(x_n) - f(c)| < \varepsilon$, as required.

(b) Suppose that $f: U \rightarrow \mathbb{R}$ is a function that does not satisfy the $\varepsilon - \delta$ criterion at c . Write down what this means and then construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to c in U for which $f(x_n)$ does not converge to $f(c)$.

Solution:

The negation of that criterion is

$$(\star) \quad \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in U \text{ s.t. } |x - c| < \delta \text{ but } |f(x) - f(c)| \geq \varepsilon.$$

We need to prove that there is a sequence $x_n \rightarrow c$ for which $f(x_n)$ does not converge to $f(c)$.

So suppose f does not satisfy the $\varepsilon - \delta$ criterion, and let $\varepsilon > 0$ be a value for which it fails.

We construct a sequence $\{x_n\}_{n \in \mathbb{N}_+}$ as follows. For each $n \in \mathbb{N}_+$, let $\delta_n = \frac{1}{n} > 0$. Then by (\star) , there is some $x \in U$ with $|x - c| < \delta_n$ for which $|f(x) - f(c)| \geq \varepsilon$. Let x_n be this element.

Now since for each n , we have $|x_n - c| < \frac{1}{n}$, we know that

$$-\frac{1}{n} \leq x_n - c \leq \frac{1}{n}$$

so by the squeeze theorem, $\lim_{n \rightarrow \infty} (x_n - c) = 0$, so $x_n \rightarrow c$. But since for every n , we have

$$|f(x_n) - f(c)| \geq \varepsilon$$

we conclude that $f(x_n)$ cannot converge to $f(c)$ (since in fact it never satisfies $|f(x_n) - f(c)| < \varepsilon$).