MAT1325: solutions to Part A of Homework #3

Question 1: matching each statement to the kind of sequence it describes.

(a) $\forall\varepsilon > 0 \exists n \in \mathbb{N}\ s.t.\ |a_n - L| < \varepsilon$. This says: no matter how I choose $\varepsilon > 0$, I can always find at least one special element of the sequence, a_n, which is no further than ε from L. Well, there are two cases to consider. If there is some $n \in \mathbb{N}$ for which $a_n = L$, this is obviously true. Otherwise, by choosing smaller and smaller values of ε, I will have to keep finding elements of the sequence closer and closer to L. Ergo: either the sequence contains L, or has a subsequence converging to L.

(b) $\forall\varepsilon > 0 \forall n \in \mathbb{N} |a_n - L| < \varepsilon$. This says: no matter how I choose $\varepsilon > 0$ and no matter how I choose $n \in \mathbb{N}$, I always have that the distance between a_n and L is less that ε. But $|a_n - L| < \varepsilon$ for EVERY $\varepsilon > 0$ can only happen if $a_n = L$; whence every element of the sequence equals L. Thus: this describes a constant sequence.

(c) $\exists N \in \mathbb{N}\ s.t.\ \forall\varepsilon > 0 |a_N - L| < \varepsilon$. This says: there is some special value of N such that no matter how you choose $\varepsilon > 0$, a_N is always at most ε from L. Again, this implies $a_N = L$ — but only for this particular value. So this describes a sequence which contains the value L.

(d) $\exists\varepsilon > 0 \text{and} \exists N \in \mathbb{N}\ s.t.\ \forall n \geq N, |a_n - L| < \varepsilon$. This says: there is a special value of $\varepsilon > 0$ and a special index N of the sequence such that every point in the sequence after N is within distance ε of L. Now saying there is some value of $\varepsilon > 0$ for which this works is not saying much; I could imagine $\varepsilon = 1000$. In fact, by taking b to be the maximum of ε and $|a_1 - L|$, $|a_2 - L|$, $\cdots$, $|a_{N-1} - L|$, you see that every element of the sequence satisfies $|a_n - L| \leq b$, whence it is a bounded sequence.

(e) $\forall\varepsilon > 0 \exists N \in \mathbb{N}\ s.t.\ \forall n \geq N, |a_n - L| < \varepsilon$ = the definition of $\lim_{n\to\infty} a_n = L$.

(f) $\exists N \in \mathbb{N}\ s.t.\ \forall\varepsilon > 0, \forall n \geq N, |a_n - L| < \varepsilon$. This says: there is a special index N such that no matter how you choose $\varepsilon > 0$, each element of the sequence after N is at a distance less than ε from L. This can only happen if $a_n = L$ for each $n \geq N$, so this describes a sequence which is eventually constant.

(g) $\exists\varepsilon > 0\ s.t.\ \forall N \in \mathbb{N} \exists n \geq N\ s.t.\ |a_n - L| < \varepsilon$. This says: there is a special value of $\varepsilon > 0$ such that no matter how you choose N, you can always find some larger value n such that a_n is within a distance ε of L. Again, since it is just one value of ε, we can imagine it is large. This time, there is no promise that every element of the sequence is close to L, just that you can keep going along the sequence, as far as you like, and you keep finding elements a_n that are kind of close to L. That is, we are describing a sequence which contains a bounded subsequence.

(h) $\forall\varepsilon > 0 \exists L' \in \mathbb{R}, \exists n \in \mathbb{N}\ s.t.\ |a_n - L'| < \varepsilon$. This says: no matter how I choose $\varepsilon > 0$, I can always find some special number L' and some element of the sequence a_n such that the distance between them is less than ε. But this is just always true; for example, I can choose $L' = a_1$.

Question 2: Putting a proof in order.

The following is a proof of the statement: Let $a \neq 0$. The function

$$f(x) = \frac{1}{x}$$

is continuous at a. Put the steps in the correct and logical order.

Proof:

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers, none of them zero, which converge to a.

We want to prove that $f(x_n) \to f(a)$.

Let $\varepsilon > 0$.

Since $a \neq 0$ and $x_n \to a$, if we set $z = \frac{1}{2}|a|$ then there is (by a DGD exercise) a value $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n| > z$.

Since $x_n \to a$, and $\varepsilon|a|z > 0$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|x_n - a| < \varepsilon|a|z$.

Set $N = \max\{N_1, N_2\}$.

Then for every $n \geq N$, we have $|f(x_n) - f(a)| = |\frac{1}{x_n} - \frac{1}{a}| = \frac{|x_n - a|}{|x_n||a|}$.

Since $n \geq N_1$, this is $\leq \frac{|x_n - a|}{z|a|}$.

Since $n \geq N_2$, this is $< \frac{\varepsilon|a|z}{z|a|}$.

Therefore, for every $n \geq N$, $|f(x_n) - f(a)| < \varepsilon$, so $f(x_n) \to f(a)$.