MAT1325 : QUIZ #1, FEBRUARY 12, 2016
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Multiple solutions are possible ; here are some.

1. (a) (2 pts) Let S be a nonempty subset of $\mathbb{R}$. Define *the supremum of S*.

   Solution: Let $s \in \mathbb{R}$. Then $s = \sup(S)$, the supremum of S, if : (a) for every $x \in S$, we have $x \leq s$; and (b) for every real number $t < s$, there is some element $x \in S$ such that $x > t$.

   Solution: The supremum of S is a real number $s \in \mathbb{R}$ such that : (a) $\forall x \in S, x \leq s$, and (b) $\forall \varepsilon > 0, \exists x \in S \; s.t. \; x > s - \varepsilon$.

   (b) (1 pt) Let $S \subset \mathbb{R}$ be a set which is bounded above, and suppose $T \subset \mathbb{R}$ is a set such that their intersection

$$S \cap T = \{x \in \mathbb{R} \mid x \in S \text{ and } x \in T\}$$

   is not empty. Prove that $S \cap T$ is bounded above.

   Solution: Let b be an upper bound of S ; then $b \geq x$ for all $x \in S$. Let $x \in S \cap T$; then $x \in S$ therefore $b \geq x$. Hence b is an upper bound of $S \cap T$.

   Solution: A set S is bounded above if there is $M \in \mathbb{R}$ such that for all $x \in S$, $x \leq M$. Here, S is bounded above so let M be an upper bound of S. Let $x \in S \cap T$. Then $x \in S$, so $x \leq M$. So M is an upper bound of $S \cap T$, so $S \cap T$ is bounded above.

   (c) (3 pts) Let S and T be two subsets of $\mathbb{R}$ which are nonempty and bounded above. Prove that if $S \cap T$ is nonempty, then

$$\sup(S \cap T) \leq \min\{\sup S, \sup T\}.$$

   (Begin by justifying the existence of the three suprema.)

   Solution: All three sets are nonempty and bounded above (using (b)), which is why the supremum exists (by the completeness axiom).
   Since $\sup(S)$ is an upper bound of S, it is an upper bound of $S \cap T$ by the proof of (b) ; same for $\sup(T)$. But $\sup(S \cap T)$ is the least upper bound of $S \cap T$, so we must have $\sup(S \cap T) \leq \sup(S)$ and $\sup(S \cap T) \leq \sup(T)$. Thus $\sup(S \cap T) \leq \min\{\sup(S), \sup(T)\}$.

   Solution: The supremum of a set exists if the set is nonempty and bounded above. By hypothesis, S and T are nonempty and bounded above, so their suprema exist. By hypothesis, $S \cap T$ is nonempty and by part (b), $S \cap T$ is bounded above, so $\sup(S \cap T)$ also exists.
   Let $x \in S \cap T$. Then $x \in S$ so $x \leq \sup(S)$, since $\sup(S)$ is an upper bound of S. In the same way, $x \leq \sup(T)$. Therefore $x \leq \min\{\sup(S), \sup(T)\}$.
   Let $t = \min\{\sup(S), \sup(T)\}$. If $\sup(S \cap T) > t$, then by the definition of $\sup(S \cap T)$, there is an element $x \in S \cap T$ such that $x > t$. But this is impossible, since for all $x \in S \cap T$, $x \leq t$. Therefore $\sup(S \cap T) \leq t$ also, whence the result.

2. (a) (2 pts) Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers. Define "the sequence $\{a_n\}_{n \in \mathbb{N}}$ converges to L."

   Solution: $\forall \varepsilon > 0 \exists N \in \mathbb{N} \; s.t. \; \forall n \geq N, |a_n - L| < \varepsilon$

   Solution: $\forall \varepsilon > 0, \exists N \in \mathbb{N} \; s.t. \; n \geq N \implies |a_n - L| < \varepsilon$

   Solution: For every positive real number ε, there is a point $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ after that point ($n \geq N$), we have $|a_n - L| < \varepsilon$.

(b) (4 pts) Prove, using the definition of convergence, that the sequence

$$a_n = \frac{5(-1)^n}{\sqrt{2n-1}-4} \qquad (n \geq 1)$$

is convergent. **Be sure to justify all steps involving inequalities or absolute values.**

Rough work : We see the numerator is either 5 or -5 and the denominator goes to ∞, so $a_n \to 0$. We will use that $|(-1)^n| = 1$ and that if $n \geq 9$ then $\sqrt{2n-1} > 4$. Then $|a_n - 0| = \frac{5}{\sqrt{2n-1}-4}$ for $n \geq 9$.

We want this to be less than ε in our proof; for which n is this true? Here are some options :

1.

$$\frac{5}{\sqrt{2n-1}-4} < \varepsilon \Leftrightarrow \frac{5}{\varepsilon} + 4 < \sqrt{2n-1} \Leftrightarrow n > \frac{1}{2}((\frac{5}{\varepsilon}+4)^2+1)$$

2. $\sqrt{2n-1} - 4 > \sqrt{n} - 4$ since $2n - 1 > n$ for $n > 1$; thus

$$\frac{5}{\sqrt{2n-1}-4} < \frac{5}{\sqrt{n}-4}$$

and this is $< \varepsilon$ if $n > (5/\varepsilon + 4)^2$.

3. If $n > 17$ then $2n - 17 > n$ (and $20 < 20\sqrt{2n-1}$ for all n) so by rationalizing,

$$\frac{5}{\sqrt{2n-1}-4} = \frac{5\sqrt{2n-1}+20}{2n-17} < \frac{5\sqrt{2n-1}+20\sqrt{2n-1}}{n} = 25\sqrt{2-\frac{1}{n}}$$

which is $< \varepsilon$ if $2 - \frac{1}{n} < \varepsilon^2/625$ or $n > 1/(2 + \varepsilon^2/625)$.

With any scenario, we will choose N to be the first positive integer for which this is true.

Solution: The limit is $L = 0$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N \geq 9$ and $N > \frac{1}{2}((\frac{5}{\varepsilon}+4)^2+1)$. Then let $n \geq N$; we have (by the rough work) $\frac{5}{\varepsilon} + 4 < \sqrt{2n-1}$ or $\frac{5}{\sqrt{2n-1}-4} < \varepsilon$. Therefore

$$\begin{aligned}
|a_n - L| &= |\frac{5(-1)^n}{\sqrt{2n-1}-4} - 0| \\
&= \frac{5}{|\sqrt{2n-1}-4|} \quad \text{since } |(-1)^n| = 1 \\
&= \frac{5}{\sqrt{2n-1}-4} \quad \text{since } n > 9 \\
&< \varepsilon \quad \text{as explained above.}
\end{aligned}$$