1. (10 pts) Solve the initial value problem using the Laplace transform:

$$y'' + 3y' + 2y = e^x, \quad y(0) = 0, \quad y'(0) = 1.$$

Denote by Y the Laplace transform of y:

$$Y = Y(s) = L\{y(x)\}(s).$$

Then

$$s^2Y - 1 + 3sY + 2Y = \frac{1}{s-1},$$

hence

$$\begin{aligned} Y = \frac{s}{(s-1)(s^2+3s+2)} &= \frac{s}{(s-1)(s+1)(s+2)} \\ &= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}, \end{aligned}$$

where A, B, C are constants determined by

$$A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) = s.$$

Substituting $s = 1$, $s = -1$, and $s = -2$ in the last equality yields

$$\begin{cases} 6A = 1 \\ -2B = -1 \\ 3C = -2 \end{cases} \Leftrightarrow \begin{cases} A = 1/6 \\ B = 1/2 \\ C = -2/3 \end{cases}$$

Thus

$$\begin{aligned} Y = \frac{1}{6} \cdot \frac{1}{s-1} + \frac{1}{2} \cdot \frac{1}{s+1} - \frac{2}{3} \cdot \frac{1}{s+2} \\ = \frac{1}{6}L\{e^x\}(s) + \frac{1}{2}L\{e^{-x}\}(s) - \frac{2}{3}L\{e^{-2x}\}(s), \end{aligned}$$

and

$$y = \frac{e^x}{6} + \frac{e^{-x}}{2} - \frac{2e^{-2x}}{3}.$$

2. (15 pts) Solve the initial value problem using the Laplace transform:

$$\ddot{x} + 2\dot{x} + x = \delta(t-2), \quad x(0) = 0, \quad \dot{x}(0) = 0.$$

Denote by X the Laplace transform of x:

$$X = X(s) = L\{x(t)\}(s).$$

Then

$$s^2X + 2sX + X = e^{-2s},$$

hence

$$\begin{aligned} X = \frac{e^{-2s}}{s^2+2s+1} &= \frac{e^{-2s}}{(s+1)^2} \\ &= e^{-2s}L\{te^{-t}\}(s) = L\{(t-2)e^{2-t}u(t-2)\}(s), \end{aligned}$$

and

$$x = (t-2)e^{2-t}u(t-2).$$

3. (10 pts) Solve the initial value problem using the Laplace transform:

$$y'' + 4y' + 4y = x^2 e^{-2x}, \quad y(0) = 0, \quad y'(0) = 0.$$

Denote by Y the Laplace transform of y:

$$y = Y(s) = L\{y(x)\}(s).$$

Then

$$s^2 Y + 4sY + 4Y = \frac{2}{(s+2)^3}.$$

Hence

$$\begin{aligned} Y = \frac{2}{(s^2+4s+4)(s+2)^3} &= \frac{2}{(s+2)^5} \\ &= \frac{1}{12} \cdot \frac{4!}{(s+2)^5} = \frac{1}{12} L\{x^4 e^{-2x}\}(s), \end{aligned}$$

and

$$y = \frac{x^4 e^{-2x}}{12}.$$

4. (15 pts) Solve the initial value problem using the Laplace transform:

$$\begin{cases} \dot{x} = 2x - 5y \\ \dot{y} = x + 4y \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$$

Denote by X and Y the Laplace transforms of x and y, respectively:

$$x = X(s) = L\{x(t)\}(s), \quad y = Y(s) = L\{y(t)\}(s).$$

Then

$$\begin{cases} sX - 1 = 2X - 5Y \\ sY = X + 4Y \end{cases} \Leftrightarrow \begin{cases} (s-2)X + 5Y = 1 \\ X - (s-4)Y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (s^2 - 6s + 13)Y = 1 \\ (s^2 - 6s + 13)X = s - 4 \end{cases}$$

Hence

$$Y = \frac{1}{s^2 - 6s + 13} = \frac{1}{(s-3-2i)(s-3+2i)} = \frac{A}{s-3+2i} + \frac{B}{s-3-2i},$$

where A, B are complex constants determined by

$$A(s-3-2i) + B(s-3+2i) = 1.$$

Substituting $s = 3 \pm 2i$ in the last equality yields $A = i/4$, $B = -i/4$. Therefore,

$$\begin{aligned} Y &= \frac{i}{4} \cdot \frac{1}{s-3+2i} - \frac{i}{4} \cdot \frac{1}{s-3-2i} \\ &= \frac{i}{4} L\{e^{(3-2i)t}\}(s) - \frac{i}{4} L\{e^{(3+2i)t}\}(s), \\ y &= \frac{ie^{(3-2i)t} - ie^{(3+2i)t}}{4} = \frac{-ie^{3t}}{4}(e^{2it} - e^{-2it}) = \frac{1}{2} e^{3t} \sin(2t). \end{aligned}$$

Similarly,

$$\begin{aligned} X &= \frac{s-4}{(s-3-2i)(s-3+2i)} = \frac{2-i}{4(s-3+2i)} + \frac{2+i}{4(s-3-2i)} \\ &= \frac{2-i}{4} L\{e^{(3-2i)t}\}(s) + \frac{2+i}{4} L\{e^{(3+2i)t}\}(s), \end{aligned}$$

$$\begin{aligned} x &= \frac{(2-i)e^{(3-2i)t} + (2+i)e^{(3+2i)t}}{4} \\ &= e^{3t} \frac{2(e^{2it} + e^{-2it}) + i(e^{2it} - e^{-2it})}{4} = e^{3t} \cos(2t) - \frac{1}{2} e^{3t} \sin(2t). \end{aligned}$$

5. (15 pts) Find and classify all equilibria of the autonomous system

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = xy - 1 \end{cases}$$

To find the equilibria, solve the system

$$\begin{cases} x - y = 0 \\ xy - 1 = 0 \end{cases} \quad \Leftrightarrow \quad x = y = 1 \text{ or } x = y = -1.$$

Thus $(1, 1)$ and $(-1, -1)$ are equilibrium points. Consider the linearized system at $(1, 1)$:

$$\begin{cases} \dot{u} = u - v \\ \dot{v} = u + v \end{cases} \quad \text{where} \quad \begin{cases} u = x - 1 \\ v = y - 1 \end{cases}$$

The poles p_1, p_2 of the Laplace transforms of u and v satisfy

$$p_1 + p_2 = 2, \quad p_1 p_2 = 2,$$

hence p_1 and p_2 are two conjugate complex numbers with positive real parts. Conclude that solutions will spiral out from the equilibrium at $(1, 1)$.

The linearized system at $(-1, -1)$ is:

$$\begin{cases} \dot{u} = u - v \\ \dot{v} = -u - v \end{cases} \quad \text{where} \quad \begin{cases} u = x + 1 \\ v = y + 1 \end{cases}$$

In this case, the poles p_1, p_2 of the Laplace transforms of u and v satisfy

$$p_1 + p_2 = 0, \quad p_1 p_2 = -2,$$

hence p_1 and p_2 are two real numbers with opposite signs. Conclude that the equilibrium at $(-1, -1)$ is a saddle point.

6. (15 pts) Let

$$y(x) = \sum_{n=0}^{\infty} y_n x^n$$

be a solution of the equation

$$(x^2 + 2)y'' + 3xy' + y = 0.$$

Determine the recurrence relation for coefficients and a lower bound for the radius of convergence of the power series $y(x)$.

Since the distance from 0 to the roots of $x^2 + 2$ in the complex plane is $|i\sqrt{2}| = \sqrt{2}$, the radius of convergence of the series $y(x)$ is at least $\sqrt{2}$.

To find the recurrence relation, note that

$$y' = \sum_{n=0}^{\infty} (n+1) y_{n+1} x^n, \quad xy' = \sum_{n=0}^{\infty} n y_n x^n,$$

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) y_{n+2} x^n, \quad x^2 y'' = \sum_{n=0}^{\infty} n(n-1) y_n x^n.$$

Substituting these expansions into the equation

$$x^2 y'' + 2y'' + 3xy' + y = 0,$$

and grouping together the coefficients of x^n leads to the relation

$$n(n-1)y_n + 2(n+1)(n+2)y_{n+2} + 3n y_n + y_n = 0, \quad n = 0, 1, 2, \ldots$$

which can be simplified as follows:

$$y_{n+2} = -\frac{n+1}{2(n+2)} y_n, \quad n = 0, 1, 2, \ldots$$