Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2384 B
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
Practise Final Exam

Instructor: Jeff Musgrave

Duration: 3 hours

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1. Solve the following initial value problems:
   (a) $(6xy^2 - 3x^2y^3 + 2y)\,dx + (9x^2y - 4x^3y^2 + 4x)\,dy = 0, \quad y(1) = 1$
   (b) $\left(x\cos^2\left(\frac{y}{x}\right) - y\right)dx + xdy = 0, \quad y(1) = 0$

2. Solve the following differential equations:
   (a) $y'' + 2y' + y = e^{-x}\ln(x)$
   (b) $y''' - 2y'' - 3y' = 6 + 10\cos(x), \quad y(0) = 3, \;\; y'(0) = -3, \;\; y''(0) = 0$
   (c) $x^2y'' - 3xy' + 4y = x$

3. Solve the following systems of differential equations
   (a)
   $$y_1' = 3y_1 - y_2 + 24t^2 - 16t,$$
   $$y_2' = y_1 + y_2 + 12$$
   (b)
   $$y_1' = 2y_1 + 3y_2,$$
   $$y_2' = 3y_1 + 2y_2$$
   $$y_1(0) = 5, \; y_2(0) = -7$$

4. Calculate the following:
   (a) $\mathcal{L}\{te^{-2t}sin(3t)\}$
   (b) $\mathcal{L}\{u(t-2)(t^2 - 7t + 3)\}$
   (c) $\mathcal{L}^{-1}\{\frac{s-16}{s^2-2s-8}\}$
   (d) $\mathcal{L}^{-1}\{\frac{2e^{-\pi s}}{s+3}\}$
   (e) $\mathcal{L}^{-1}\{\frac{1}{s^2-s-12}\}$ (use convolution)

5. Use the Laplace transform to solve the following initial value problems:
   (a) $y'' - 4y' - 5y = \delta(t-3), \;\; y(0) = 4, \;\; y'(0) = 2$
   (b) $y'' + y = t - u(t-1)t, \;\; y(0) = 0, \;\; y'(0) = 0$

6. (a) Use Simpson's rule with $n = 8$ to approximate the solution of $\int_1^2 \frac{2}{1+x}\,dx$ to six decimal places. Compare your approximations with the true values by calculating the simple error.

   (b) Use Gaussian quadrature with three steps to approximate the solution of $\int_1^2 \frac{2}{1+x}\,dx$ to six decimal places. Compare your approximations with the true values by calculating the simple error.

7. Use the Runge-Kutta Method of order 4 with $h = 0.5$ to calculate (to 4 decimal places) the first two steps (*ie* y_1 and y_2) of the numerical solution of $y' = 2xe^{-y}$, $y(0) = 0$ on $[0, 1]$. Compare the approximations with the true values by calculating the simple error.

Formulas

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	
t^n	$n!/s^{n+1}$	; $n = 0, 1, 2, \ldots$ and $s > 0$
e^{at}	$1/(s-a)$	; $s > a$
$\sin(kt)$	$k/(s^2+k^2)$	; $s > 0$
$\cos(kt)$	$s/(s^2+k^2)$	; $s > 0$
$\sinh(kt)$	$k/(s^2-k^2)$	; $s > k$
$\cosh(kt)$	$s/(s^2-k^2)$	; $s > k$
$\delta(t-a)$	e^{-as}	; $s > 0$
$u(t-a)$	$\frac{e^{-as}}{s}$	; $s > 0$

$$
\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st} f(t)dt \\
\mathcal{L}\{e^{at} f(t)\} &= F(s-a) \\
\mathcal{L}\{u(t-a)f(t-a)\} &= e^{-as}F(s) \\
\mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^n}{ds^n} F(s) \\
\mathcal{L}\left\{\int_0^t f(x)dx\right\} &= \frac{1}{s}F(s) \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty F(x)dx \\
\mathcal{L}\left\{\frac{d^n}{dt^n} f(t)\right\} &= s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) \\
(f * g)(t) &= \int_0^t f(x)g(t-x)dx \\
\mathcal{L}\{(f*g)(t)\} &= F(s)\,G(s)
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b f(x)dx &= h\sum_{j=1}^{n} f(x_j^*)\,, \qquad |\epsilon| \le \frac{1}{24}\,M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\,|f''(x)| \\
\int_a^b f(x)dx &= \frac{h}{2}\sum_{j=1}^{n} \left(f(x_{j-1}) + f(x_j)\right), \qquad |\epsilon| \le \frac{1}{12}\,M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\,|f''(x)| \\
\int_a^b f(x)dx &= \frac{h}{3}\sum_{j=0}^{n-1} \left(f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})\right), \qquad |\epsilon| \le \frac{1}{180}\,M\,(b-a)\,h^4\,, \\
& \quad M = max_{a\le x\le b}\,|f^{(4)}(x)|
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= h f(x_n, y_x) \\
k_2 &= h f(x_{n+1}, y^*_{n+1}) = h f(x_{n+1}, y_n + k_1) \\
y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2)
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= h\, f(x_n, y_n) \\
k_2 &= h\, f(x_n + \frac{1}{2}\, h, y_n + \frac{1}{2}\, k_1) \\
k_3 &= h\, f(x_n + \frac{1}{2}\, h, y_n + \frac{1}{2}\, k_2) \\
k_4 &= h\, f(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + \frac{1}{6}\,(k_1 + 2\, k_2 + 2\, k_3 + k_4)
\end{aligned}
$$

$$
\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} A_i f(t_i)
$$

Order n	Nodes t_i	Coefficients A_i
2	-0.5773502692	1.0
	0.5773502692	1.0
3	-0.7745966692	0.555555556
	0.0	0.888888889
	0.7745966692	0.555555556
4	-0.8611363116	0.3478548451
	-0.3399810436	0.6521451549
	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	-0.9061798459	0.2369268850
	-0.5384693101	0.4786286705
	0.0	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850