1. Let $A \subseteq \mathbf{R}$.

MAT 2125 W13 Final.

2 a) If $s \in \mathbf{R}$, define what is meant by "s is the supremum of A", i.e. $s = \sup A$.

2 b) If $l \in \mathbf{R}$, define what is meant by "l is the infimum of A", i.e. $l = \inf A$.

2 c) State necessary and sufficient conditions for $\sup A$ to exist.

Now suppose both $l = \inf A$, and $s = \sup A$ exist.

4 d) Prove that $s - l = \sup\{\, a - a' \mid a, a' \in A\}$.

a) See your notes : $s = \sup A$ if 1) $\forall a \in A$, $a \leq s$, & 2) $\forall \varepsilon > 0\ \exists\, a \in A$ with $s - \varepsilon < a \leq s$

b) see your notes : $l = \inf A$ if 1) $\forall a \in A$, $l \leq a$, & 2) $\forall \varepsilon > 0\ \exists\, a \in A$ s.t. $l \leq a < l + \varepsilon$.

c) The supremum of A exists iff A is nonempty and bounded above.

d) We first show $s - l$ is an upper bound for $B = \{a - a' \mid a, a' \in A\}$. Let $a - a' \in B$. Then $a \leq s$ and $a' \geq l$, so $a - a' \leq s - l$. Since a, a' were arbitrary, $s - l$ is an upper bound for B. 2

Now, let $\varepsilon > 0$. As $s = \sup A$, $\exists\, a \in A$ s.t. $s - \frac{\varepsilon}{2} < a \leq s$, and as $l = \inf A$, $\exists\, a' \in A$ s.t. $l \leq a' < l + \frac{\varepsilon}{2}$. Then $-l \geq -a' > -l - \frac{\varepsilon}{2}$, so $s - \frac{\varepsilon}{2} + (-l - \frac{\varepsilon}{2}) < a + a' \leq s - l$; i.e. $(s - l) - \varepsilon < a + a' \leq s - l$. Hence

2

$s - l = \sup B$.

2. Let $\{a_n\}_{n\geq 1}$ be a real sequence.

2 a) Define "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence."

1 b) Define "$\{a_n\}_{n\geq 1}$ is a bounded sequence."

1 c) State the triangle inequality for $\mathbf{R}$.

3 d) Prove, from first principles, that is, directly from the definitions in (a) and (b), and (c), if necessary, that a Cauchy sequence is bounded.

3 e) Give an example of a Cauchy sequence $\{a_n\}_{n\geq 1}$ with $a_n > 0$, $\forall n \geq 1$ but where $\left\{\frac{1}{a_n}\right\}_{n\geq 1}$ is *not* Cauchy. You must justify your answer.

a) $\{a_n\}_{n\geq 1}$ is Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall m,n \geq N$, $|a_n - a_m| < \varepsilon$.

b) $\{a_n\}_{n\geq 1}$ is bounded if $\exists M$ s.t. $\forall n \geq 1$, $|a_n| \leq M$.
($\Leftrightarrow \exists M'$ s.t. $\forall n \geq 1$, $|a_n| < M'$).

c) $\forall x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.

d) Let $\{a_n\}_{n\geq 1}$ be Cauchy and let $\varepsilon = 1$. Then, $\exists N \in \mathbb{N}$ st $\forall m,n \geq N$, $|a_n - a_m| < \varepsilon = 1$. In particular, $\forall n \geq N$, $|a_n - a_N| < 1$. By the Δ inequality, $\forall n \geq N$, $|a_n| \leq |a_n - a_N| + |a_N| \leq 1 + |a_N|$. 1
Now set $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1+|a_N|\}$. Then, 1
$\forall n \geq 1$, $|a_n| \leq M$. Hence $\{a_n\}_{n\geq 1}$ is bdd.

e) Let $a_n = \frac{1}{n}$. Then $a_n \to 0$ and so $\{a_n\}_{n\geq 1}$ is Cauchy (or, directly: Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ st $N > \frac{1}{\varepsilon}$. Then, $\forall n \geq N$, ~~$\forall k \geq 1$~~,
$|a_n - a_{n+k}| = \frac{k}{n(n+k)} = \frac{1}{n(\frac{n}{k}+1)} < \frac{1}{n} < \varepsilon$.)

However, $\frac{1}{a_n} = n$, and the sequence $\{n\}_{n\geq 1}$ is not bdd, and so is not Cauchy. ($\forall M > 0$, $\exists n \in \mathbb{N}$ s.t. $n \geq M$ (Archimedean prop) so no $M \in \mathbb{R}$ is an upper bound for $\{n\}_{n\geq 1}$).

1 - correct e.g
1 - pf Cauchy
1 - pf $\frac{1}{a_n}$ not Cauchy

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

2

$$\text{“The series } \sum_{n=1}^{\infty} c_n \text{ converges.”}$$

Now consider the two series

$$A: \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$B: \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+1}}$$

3 b) Does the series A converge?

3 c) Does the series B converge?

2 d) Does the series B converge absolutely?

(In (b)–(d), you may use known tests and theorems, but be sure to verify their hypotheses.)

a) The series $\sum_{n=1}^{\infty} c_n$ converges if the sequence $\left\{ \sum_{k=1}^{n} c_k \right\}_{n\geq 1}$ converges.

b) Note that $0 \leq \frac{n!}{n^n} = \frac{n(n-1)\cdots 2 \cdot 1}{n(n)\cdots n\, n} \leq \frac{2}{n^2}$ if $n \geq 2$. Since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{n!}{n^n}$, by comparison.

c) This is an alternating series, so we see if Leibniz thm applies:
clearly, $n+1 > n \Rightarrow (n+1)^2+1 > n^2+1 \Rightarrow \sqrt{(n+1)^2+1} > \sqrt{n^2+1} \Rightarrow \left\{ \frac{1}{\sqrt{n^2+1}} \right\}_1$ is decreasing. Moreover, $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 0 \cdot 1 = 0$, so indeed Leibniz' thm applies to show that the series B does converge.

d) To see that B does not converge absolutely, we note that

$$\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{(n+1)^2}} = \frac{1}{n+1}, \text{ and } \sum_{n=1}^{\infty} \frac{1}{n+1} \text{ does not converge,}$$

(Since its partial sum $S_n = ($partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}) - 1$.)

$\frac{1}{2}$ - correct
concl

4. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$. 4 40?

2 a) Define

"The function f is continuous at a."

2 b) Prove that $\forall x \in \mathbf{R}$, $|x| < \frac{1}{2} \Longrightarrow |1 - x^2| > \frac{3}{4}$.

6 c) Define $f : \mathbf{R} \setminus \{-1, 1\} \to \mathbf{R}$ by $f(x) = \dfrac{1}{1 - x^2}$. Prove carefully that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x \in \mathbf{R} \setminus \{-1, 1\} \text{ and } |x| < \delta \quad \Longrightarrow \quad |f(x) - 1| < \varepsilon.$$

a) The f^n f is cts at a if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $x \in A$ and $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

b) Note that $|1| \leq |1 - x^2| + |x^2|$, so $|1 - x^2| \geq 1 - x^2 > 1 - \frac{1}{4} = \frac{3}{4}$, if $|x| < \frac{1}{2}$.

for $x \in \mathbb{R} \setminus \{1, -1\}$,

c) Note that $|f(x) - 1| = \left| \frac{1}{1 - x^2} - 1 \right| = \left| \frac{1 - (1 - x^2)}{1 - x^2} \right| = \frac{x^2}{|1 - x^2|}$. 1

Let $\varepsilon > 0$ and choose $\delta = \min\left(\frac{1}{2}, \frac{\sqrt{3 \cdot \varepsilon}}{2}\right)$. Then, $x \in \mathbb{R} \setminus \{1, -1\}$ and

$|x| < \delta \Rightarrow |x| < \frac{1}{2}$ so $\frac{x^2}{|1 - x^2|} < \frac{4}{3} \cdot x^2 < \frac{4}{3} \cdot \delta^2 \leq \frac{4}{3} \frac{3}{4} \cdot \varepsilon = \varepsilon$.

1

i.e. $x \in \mathbb{R} \setminus \{1, -1\}$ and $|x| < \delta \Rightarrow |f(x) - 1| < \varepsilon$.

$|1 - x^2| > \frac{3}{4}$

1

Or $\delta = \min\left(\frac{1}{2}, \frac{3\varepsilon}{2}\right)$.

good δ

good pf 2

$\delta < \frac{1}{2}$

Good δ 2

$\frac{2 \cancel{4}}{3} \frac{1}{\cancel{2}} \cdot \delta < \varepsilon$

$\frac{4}{3} \delta^2 < \varepsilon \quad \delta < \sqrt{\frac{3\varepsilon}{4}} = \frac{\sqrt{3}}{2}\varepsilon$ Good pf 1

5. Let A and B be subsets of $\mathbf{R}^2$, and suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function., $\mathcal{U} \subseteq \mathbb{R}$ is open

2 a) Define "A is open".

2 b) Define "B is compact".

1 c) State a sufficient condition for A to be open, in terms of the function f and $\mathcal{U}$.

Now define $C = \{(x, y) \in \mathbf{R}^2 \mid xy > 0\}$, and $D = \{(x, y) \in \mathbf{R}^2 \mid xy = 0\}$.

3 d) Prove that C is open.

2 e) Prove that D is closed, but is not compact.

a) A is open if $\forall a \in A$, $\exists r > 0$ s.t. $B(a, r) \subseteq A$.

b) B is compact if every sequence in B has an accumulation point in B. $1\frac{1}{2}/2$ for cld & bdd

c) A would be open if $A = f^{-1}(\mathcal{U})$, for some open set $\mathcal{U} \subseteq \mathbb{R}$.

d) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = xy$. As a product of cts fn, g is cts. Noting that $C = g^{-1}((0,\infty))$, and the fact that $(0,\infty)$ is open in $\mathbb{R}$ shows that A is open.

e) For the function g of (d), we note that $B = g^{-1}(\{0\})$, and that $\{0\}$ is closed in $\mathbb{R}$. Hence B is closed. B is not compact because B is not bounded (it contains the points $\{(0,n) \mid n \in \mathbb{N}\}$, for example.).

$1 + 1$

$D = C^c \cup -C^c$ $\quad 1\frac{1}{2}$

+ pf not bdd

6. Let $A \subset \mathbf{R}^n$, $b \in \mathbf{R}^n$, and suppose $f : A \to \mathbf{R}^p$ is f is uniformly continuous on A.

2 a) Define "b is a limit point of A", without referring to sequences in A.

2 b) Give a characterization of limit points in terms of sequences $\{a_n\}_{n\geq 1} \subseteq A$.

2 c) Define "f is uniformly continuous on A."

4 d) Prove that if $\{a_n\}_{n\geq 1} \subseteq A$ and $a_n \to b \in \mathbf{R}^n$, then $\{f(a_n)\}_{n\geq 1}$ is Cauchy. (Note that you may not assume that $b \in A$.)

a) The point b is a limit point of A if $\forall r > 0$, $B(b, r) \cap A \neq \emptyset$.

b) The point b is a limit point of A if $\exists$ sequence $\{a_n\}_{n\geq 1} \subseteq A$ s.t. $a_n \to b$.

c) The fn f is uniformly cts on A if
$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall x, y \in A$, $\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$

d) Since $\{a_n\}_{n\geq 1}$ is convergent, it is Cauchy. 1 So let $\varepsilon > 0$. By the uniform cty of f on A, $\exists \delta > 0$ s.t. $\forall x, y \in A$, $\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$. For $\delta > 0$, as $\{a_n\}_{n\geq 1}$ is Cauchy, $\exists N \in \mathbb{N}$ st. $\forall n, p \geq N$, $\|a_n - a_p\| < \delta$. Thus,

$\forall n, p \geq N$, $\|f(a_n) - f(a_p)\| < \varepsilon$. Hence $\{f(a_n)\}_{n\geq 1}$ is Cauchy. 1

2 - "Cauchy + Cauchy"

2 - $\varepsilon \to \delta \to N$
1 1

1/4 - "cts at b"

4

7. Define $f:[0,1]\to\mathbf{R}$ by

$$f(x)=\begin{cases}0 & \text{for } x\in[0,\frac{1}{2}]\\ 1 & \text{for } x\in(\frac{1}{2},1]\end{cases}$$

3 a) State carefully a necessary and sufficient condition for f to be integrable in terms of upper sums $U(f,P)$ and lowers sums $L(f,P)$, where P denotes a partition of $[0,1]$. (You may give the defintion or an equivalent condition.)

5 b) For $n\geq 3$ (!), $n\in\mathbf{N}$, let P_n be the partition $P_n=\{0,\frac{1}{2},\frac{1}{2}+\frac{1}{n},1\}$. Find $U(f,P_n)$ 2 and $L(f,P_n)$ 2.

2 c) Use your result in (b) and your response in (a) to prove directly that f is integrable.

a) The (bounded) function f is integrable on $[0,1]$ if $\forall\,\varepsilon>0$, $\exists$ partition P of $[0,1]$ s.t. $U(f,P)-L(f,P)<\varepsilon$.

1 Note that $\forall x\in[0,1]$, $0\leq f(x)\leq 1$.

Also note that for P_n: $x_0=0$, $x_1=\frac{1}{2}$, $x_2=\frac{1}{2}+\frac{1}{n}$ and $x_3=1$.

Thus, $m_1(f)=\inf\{f(x)\mid x\in[0,\frac{1}{2}]\}=0$, by def^n of f;

1 $m_2(f)=\inf\{f(x)\mid x\in[\frac{1}{2},\frac{1}{2}+\frac{1}{n}]\}=0$, since $f(\frac{1}{2})=0$;

$m_3(f)=\inf\{f(x)\mid x\in[\frac{1}{2}+\frac{1}{n},1]\}=1$, by def^n of f.

Thus 1 $\underline{L(f,P_n)}=0\cdot\frac{1}{2}+0\cdot\frac{1}{n}+1\cdot\left(\frac{1}{2}-\frac{1}{n}\right)=\underline{\frac{1}{2}-\frac{1}{n}}$.

On the other hand, noting that $0\leq f(x)\leq 1$,

$M_1(f)=\sup\{f(x)\mid x\in[0,\frac{1}{2}]\}=0$, since $f\equiv 0$ here;

1 - each correct answer

$M_2(f)=\sup\{f(x)\mid x\in[\frac{1}{2},\frac{1}{2}+\frac{1}{n}]\}=1$, since $f(\frac{1}{2}+\frac{1}{n})=1$;

1 - def^n.

$M_3(f)=\sup\{f(x)\mid x\in[\frac{1}{2}+\frac{1}{n},1]\}=1$, since $f\equiv 1$ here.

1 - "sup" & "inf"

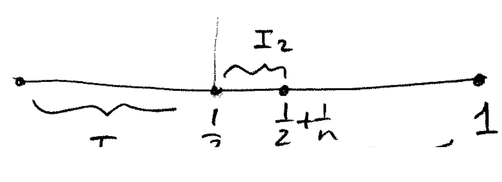
thus $U(f,P_n)=0\cdot\frac{1}{2}+1\cdot\frac{1}{n}+1\cdot\left(\frac{1}{2}-\frac{1}{n}\right)=\frac{1}{n}+\frac{1}{2}-\frac{1}{n}=\frac{1}{2}$.

$=\frac{1}{2}$

c) Let $\varepsilon>0$ and choose $n>\frac{1}{\varepsilon}$. Then,

$U(f,P_n)-L(f,P_n)=\frac{1}{n}<\varepsilon$. Hence,

f is integrable on $[0,1]$.



8. Define a function $f : [0, \frac{1}{2}] \to \mathbf{R}$ by by $f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt$.

2 a) ***Briefly*** **explain why f is continuous on $[0, \frac{1}{2}]$ and differentiable on $(0, \frac{1}{2})$. (Use theorems!)**

2 b) **State your favourite version of the Mean Value Theorem (for *derivatives*).**

2 c) **Prove that f is strictly increasing on $[0, \frac{1}{2}]$.**

Now denote $f(\frac{1}{2}) = p$. Let

$$s : [0, p] \to [0, \frac{1}{2}]$$

be the inverse function for f, which we know exists by parts (a) and (c), and define $c : [0, p] \to \mathbf{R}$ by

$$c(x) = \sqrt{1 - s^2(x)}.$$

4 d) ***Briefly*** **explain why s and c are differentiable (use theorems) and show that $s'(x) = c(x)$, and $c'(x) = -s(x)$ for all $x \in [0, p]$.**

1

a) The fn $g(t) = \frac{1}{\sqrt{1-t^2}}$ is cts on $[0, \frac{1}{2}]$ because $x \mapsto \sqrt{x}$ is cts on $(0, \infty)$, and $1 - t^2 > 0$ on $[0, \frac{1}{2}]$. Hence, g is integrable and so f is cts.

1/2 As g is cts on $[0, \frac{1}{2}]$, the FTC guarantees f is diff'ble on $(0, \frac{1}{2})$

b) If f is cts on $[a, b]$ and diff'ble on (a, b), $\exists\, \xi \in (a, b)$ s.t.

$$f(b) - f(a) = f'(\xi)(b - a).$$

c) If $x, y \in [0, \frac{1}{2}]$ and $y < x$, then by (b),

$f(y) - f(x) = f'(\xi)\underbrace{(y - x)}_{>0}$ for some $\xi \in (x, y)$. But $f'(\xi) = \frac{1}{\sqrt{1-\xi^2}}$

on $(0, \frac{1}{2})$, and so $f(y) - f(x) > 0$. Thus f is strictly increasing on $[0, \frac{1}{2}]$.

d) Since f is diff'ble on $(0, \frac{1}{2})$ and has a non-zero derivative, its inverse s is on $f(0, \frac{1}{2}) = (0, p)$, by (c). Moreover, $s'(x) = \frac{1}{f'(s(x))} = \frac{1}{\frac{1}{\sqrt{1 - s^2(x)}}} = \sqrt{1 - s^2(x)} = c(x)$

by defn of c. Since $1 - s^2(x) > 0$ $\forall x \in (0, p)$ (as $s(x) \in (0, \frac{1}{2})$, $\forall x \in (0, p)$)

and $t \mapsto \sqrt{t}$ is diff'b on $(0, \infty)$, the fn c is also diff'ble on $(0, p)$.

Moreover, $c'(x) = \frac{1}{2} \frac{(-2s(x)s'(x))}{\sqrt{1 - s^2(x)}} = -\frac{s(x)\,c(x)}{c(x)} = -s(x)$.

2 **9.** a) Suppose $f_n : [a,b] \to \mathbf{R}, n \geq 1$ is a sequence of functions, and $f : [a,b] \to \mathbf{R}$. Define what is meant by

$$\sum_{n=1}^{\infty} f_n \quad \textit{converges uniformly to } f \text{ on } [a,b].$$

b) Define $h : \mathbf{R} \setminus \{-1\} \to \mathbf{R}$ by $h(x) = \dfrac{1}{1+x}$.

1 (i) Show that $h^{(n)}(0) = n!(-1)^n$

2 (ii) Give the Taylor polynomial of h of order 3 at 0, with remainder term

2 (iii) Prove that the Taylor series for h at 0 converges uniformly on any closed interval $[-r,r]$, $r < 1$.

$1\frac{1}{2}$ (iv) Prove that the Taylor series for h at 0 does not converge to h at $x = +1$, even though $h(+1) = \frac{1}{2}$.

2 (v) Prove that the Taylor series for h at 0 converges to h on $(-1,1)$.

a) The series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a,b]$ if the sequence $\left\{\sum_{k=1}^{n} f_k\right\}_{n\geq 1}$ converges uniformly to f on $[a,b]$, i.e. $\forall \varepsilon > 0\ \exists N$ st $\forall n \geq N$, $\forall x \in [a,b]$, $\left|\sum_{k=1}^{n} f_k(x) - f(x)\right| < \varepsilon$.

b) (i) Since $h(x) = (1+x)^{-1}$, we claim, $\forall n \in \mathbb{N}$, $h^{(n)}(x) = (-1)^n n! (1+x)^{-n-1}$ $(*)$ This is clearly true for $n = 0$, so assume $h^{(n)}(x) = (-1)^n (1+x)^{-n-1}$. Then, $h^{(n+1)}(x)$

$= h^{(n)}(x)' = (-1)^n n!(-n-1)(1+x)^{-n-2} = (-1)^{n+1}(n+1)!\,(1+x)^{-(n+1)-1}$.

Thus, by induction, $(*)$ holds. Hence $h^{n}(0) = (-1)^n n!$

(ii) $p(x) = h(0) + h'(0)x + \dfrac{h''(0)x^2}{2} + \dfrac{h^{(3)}(0)\,x^3}{6}$

$= 1 - x + x^2 - x^3$, 1

and $R_3(x) = \dfrac{h^{(4)}(\xi)\,x^4}{4!} = \dfrac{x^4}{(1+\xi)^5}$ for some ξ between 0 & x. 1

(iii) Indeed, the Taylor series for h at 0 is $\sum_{n=0}^{\infty} (-1)^n x^n$, which is a geometric series. If $|x| \leq r < 1$, then as $\sum_{n=0}^{\infty} r^n$ converges, and $|(-1)^n x^n| \leq r^n$, by the Weierstrass M-test, the series converges uniformly on $[-r,r]$ (if $r<1$.)

9(c) (iv) The Taylor series for ln at 0 for $x=1$ is $\sum_{n=0}^{\infty} (-1)^n$, which does not converge because the terms $(-1)^n$ do not tend to 0 as $n \to \infty$.

(v) We know $\sum_{k=0}^{n} (-x)^k = \dfrac{1-(-x)^{n+1}}{1-(-x)}$, and

if $|x|<1$, $|-x|<1$ so $(-x)^{n+1} \to 0$ as $n \to \infty$.

Hence $\sum_{k=0}^{\infty} (-x)^k = \lim_{n\to\infty} \sum_{k=0}^{\infty} (-x)^k = \lim_{n\to\infty} \dfrac{1-(-x)^{n+1}}{1-(-x)}$

$$= \frac{1}{1+x} - \ln(x).$$

---

10 - See P. 224 Example 5.7 of your text, although the proof there is too terse to earn 100% on the bonus question – there are gaps.
