MATH3705 A — Test 1: 5:35pm–6:25pm, May 29

Name and Student Number:

Total points: 15. No partial marks for Questions 1-4.

Closed book! Formula Sheet and Non-programmer calculators are allowed!

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[2] 1. $L\{t \sin 3t\} =$

(a) $\frac{3s}{(s^2+3)^2}$ (b) $\frac{3s}{s^2+9}$ (c) $\frac{2\sin s}{(s^2+9)^2}$ (d) $-\frac{2e^s}{(s^2+3)^2}$ (e) $\frac{6s}{(s^2+9)^2}$

Solution: (e)

$L\{\sin 3t\} = \frac{3}{s^2+9}, \Rightarrow L\{t \sin 3t\} = -\left(\frac{3}{s^2+9}\right)' = \frac{6s}{(s^2+9)^2}.$

[2] 2. Find $L\{f(t)\}$, where $f(t) = 3t^2 - e^{-3t}\cos(4t)$.

(a) $\frac{12}{s^2} - \frac{(s+3)}{(s+3)^2+4}$ (b) $\frac{6}{s^3} - \frac{(s-3)}{(s-3)^2+16}$ (c) $\frac{12}{s^3} - \frac{(s-3)}{(s-3)^2+16}$
(d) $\frac{6}{s^3} - \frac{(s+3)}{(s+3)^2+16}$ (e) $\frac{3}{s^3} - \frac{(s+3)}{(s+3)^2+16}$

Solution: (d)

By linearity of LT and the First Shift Theorem, we have

$$F(s) = \frac{3(2!)}{s^3} - \frac{(s+3)}{(s+3)^2+16}.$$

[2] 3. Let $f(t)$ be 2-periodic for $t \geq 0$, and $f(t) = \begin{cases} 0, & 0 \leq t < 1; \\ 1, & 1 \leq t < 2. \end{cases}$ Find $L\{f(t)\}$.

(a) $\frac{-s(e^{-s}-e^{-2s})}{1-e^{-2s}}$ (b) $\frac{e^{-s}+e^{-2s}}{s(1-e^{-2s})}$ (c) $\frac{s(e^{-s}-e^{-2s})}{1-e^{-2s}}$ (d) $\frac{-e^{-s}+e^{-2s}}{s(1-e^{-2s})}$ (e) $\frac{e^{-s}-e^{-2s}}{s(1-e^{-2s})}$

Solution: (e)

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt = \frac{1}{1-e^{-2s}}\int_1^2 e^{-st}dt$$
$$= \frac{1}{1-e^{-2s}}\frac{e^{-st}}{-s}\Big|_1^2 = \frac{e^{-s}-e^{-2s}}{s(1-e^{-2s})}.$$

[2] 4. Calculate $L\{4t\sqrt{t}\}$.

(a) $\frac{3\sqrt{\pi}}{s^{5/2}}$ (b) $\frac{3\pi}{s^{5/2}}$ (c) $\frac{3}{\sqrt{\pi}s^{5/2}}$ (d) $\frac{3\sqrt{\pi}}{s^{5/2}}$ (e) $\frac{\pi}{s^{5/2}}$

Solution: (a), or (d):

$$\begin{aligned} L\{4t\sqrt{t}\} = L\{4t^{\frac{3}{2}}\} &= \frac{4\Gamma(\frac{5}{2})}{s^{5/2}} \\ &= \frac{3\sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{s^{5/2}}. \end{aligned}$$

[4] 5. Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = 70e^{4t}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: Applying Laplace transform to the two sides, we get the subsidiary equation

$$s^2F + sF - 6F = \frac{70}{s-4}$$

or

$$F(s) = \frac{70}{(s-4)(s+3)(s-2)}$$

By partial fraction

$$\begin{aligned} \frac{70}{(s-4)(s+3)(s-2)} &= \frac{A}{s-4} + \frac{B}{s+3} + \frac{C}{s-2} \\ 70 &= A(s+3)(s-2) + B(s-4)(s-2) + C(s+3)(s-4). \end{aligned}$$

To find A, B, C, we use special s values: $s = 4$ gives $A = 5$, and $s = -3$ gives $B = 2$, and $s = 2$ gives $C = -7$. Putting all this together we have

$$F(s) = \frac{5}{s-4} + \frac{2}{s+3} + \frac{-7}{s-2},$$

Thus

$$f(t) = 5e^{4t} + 2e^{-3t} - 7e^{2t}.$$

[**3**] 6. Find $L^{-1}\left\{\frac{-5e^{-3s}}{s(s^2+4s+5)}\right\}$.

Solution: Let $G(s) = \frac{-5}{s(s^2+4s+5)}$. By partial fraction, we have

$$G(s) = \frac{-1}{s} + \frac{s+4}{s^2+4s+5} = -\frac{1}{s} + \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1}.$$

Thus

$$g(t) = -1 + e^{-2t}\cos(t) + 2e^{-2t}\sin(t).$$

By the Second Shift Theorem,

$$f(t) = u(t-3)\left[-1 + e^{-2(t-3)}\cos(t-3) + 2e^{-2(t-3)}\sin(t-3)\right].$$