Assignment 1 Solutions

1/ a) i) $$\frac{f(5)-f(0)}{5-0} = \frac{\sqrt{5^2+1}-\sqrt{0^2+1}}{5} = \frac{\sqrt{26}-\sqrt{1}}{5} \approx 0.82$$

ii) $$\frac{f(10)-f(5)}{10-5} = \frac{\sqrt{10^2+1}-\sqrt{5^2+1}}{5} = \frac{\sqrt{101}-\sqrt{26}}{5} \approx 0.99$$

iii) $$\frac{f(5)-f(4)}{5-4} = \frac{\sqrt{5^2+1}-\sqrt{4^2+1}}{1} = \frac{\sqrt{26}-\sqrt{17}}{1} \approx 0.98$$

iv) $$\frac{f(6)-f(5)}{6-5} = \frac{\sqrt{6^2+1}-\sqrt{5^2+1}}{1} = \frac{\sqrt{37}-\sqrt{26}}{1} \approx 0.98$$

b) $$\frac{f(5)-f(4.5)}{5-4.5} = \frac{\sqrt{5^2+1}-\sqrt{4.5^2+1}}{0.5} = \frac{\sqrt{26}-\sqrt{21.25}}{0.5} \approx 0.98$$

$$\frac{f(5.5)-f(5)}{5.5-5} = \frac{\sqrt{5.5^2+1}-\sqrt{5^2+1}}{0.5} = \frac{\sqrt{31.25}-\sqrt{26}}{0.5} \approx 0.98$$

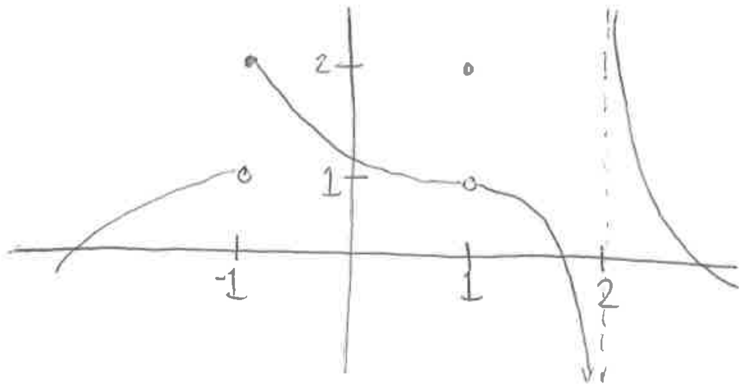
From these estimates, 0.98 is a reasonable approximation of the instantaneous rate of change of $f(x)=\sqrt{x^2+1}$ at $x=5$.

2/ a) $\lim_{x\to 5} 6 = 6$

b) $\lim_{x\to 5} x^3-2x+3 = 5^3-2(5)+3 = 118$

c) $\lim_{x\to 1^-} \sqrt{x-1}$ does not exist, since for any $x<1$, $x-1$ is negative, in which case the expression $\sqrt{x-1}$ is undefined.

3/



Say that this graph represents a function, $f(x)$. Then $f(x)$ is discontinuous at $x=-1$ because $\lim_{x\to -1^-} f(x) = 1 \neq 2 = \lim_{x\to -1^+} f(x)$, so that $\lim_{x\to -1} f(x)$ does not exist.

$f(x)$ is discontinuous at $x=1$ since $\lim_{x\to 1} f(x) = 1 \neq 2 = f(1)$.

It is discontinuous at $x=2$ since $f(2)$ is undefined.

4/ $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$

$= \lim_{x\to 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)}$

$= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)}$

$= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2}$ ← (Here is where said fact is used)

$= \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

5/ $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{(x+h)^3+1}}{3(x+h)+2} - \frac{\sqrt{x^3+1}}{3x+2}}{h}$

$f'(3) = \lim_{h\to 0} \frac{\frac{\sqrt{(3+h)^3+1}}{3(3+h)+2} - \frac{\sqrt{3^3+1}}{3(3)+2}}{h}$

6/ $f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$

$= \lim_{h \to 0} \frac{(4+h)^2 - 1 - (4^2 - 1)}{h}$

$= \lim_{h \to 0} \frac{16 + 8h + h^2 - 1 - 16 + 1}{h}$

$= \lim_{h \to 0} \frac{h(8+h)}{h} = \lim_{h \to 0} 8 + h = 8$